Ellipsoidal path connections for Time-of-Flight Rendering (Supplementary material)

March 30, 2019

1 Intersection of ellipsoid and plane

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

$$p' D p = 1$$

Note that the ellipsoid in our paper will constrained to $c = b$, $a = \frac{\tau}{2}$, $b = \sqrt{a^2 - \frac{\|f_1 - f_2\|^2}{2}} = a\sqrt{1 - k^2}$, where k is eccentricity.

Let
C_1, C_2, C_3 be the corners of the triangle and
$T = \frac{C_2 - C_1}{\|C_2 - C_1\|}$; $\hat{n} = \frac{(C_2 - C_1) \times (C_3 - C_1)}{\|(C_2 - C_1) \times (C_3 - C_1)\|}$; $U = \hat{n} \times \hat{T}$
$O =$ center of ellipse in T-U Plane (unknown)
Equation of the plane $p = O + tT + uU = \begin{bmatrix} T & U & O \end{bmatrix} \begin{bmatrix} t \\ u \\ 1 \end{bmatrix} = Ev$

At the intersection of the ellipsoid and plane containing the triangle, we have $v'E'DEv = 1$

$$\begin{bmatrix} t & u & 1 \end{bmatrix} \begin{bmatrix} \langle T, T \rangle_D & \langle T, U \rangle_D & \langle T, O \rangle_D \\ \langle U, T \rangle_D & \langle U, U \rangle_D & \langle U, O \rangle_D \\ \langle O, T \rangle_D & \langle O, U \rangle_D & \langle O, O \rangle_D - 1 \end{bmatrix} \begin{bmatrix} t \\ u \\ 1 \end{bmatrix} = 0,$$

Which is in $T - U$ coordinate system.

Let us consider the ellipse through origin in T-U plane orientated at an angle θ with T-axis (in clockwise direction) and m_1 and m_2 as major and minor axis respectively.
$\frac{(t \cos\theta - u \sin\theta)^2}{m_1^2} + \frac{(t \sin\theta + u \cos\theta)^2}{m_2^2} = 1$ coefficient of t, u are zero.

Hence, $\langle T, O \rangle_D = 0$ and $\langle U, O \rangle_D = 0$. Also using the fact that the C_1, C_2, C_3, O are co-planar, we get one more equation. To compute O, we have three equations and three unknowns (O_x, O_y, O_z) and can solve for O. However, we will use a simpler algorithm described in algorithm 1 to compute the O. Notice that $\langle O, O \rangle_D - 1 < 0$. If not, the ellipsoid do not intersect the triangle. This is one of the early triangle rejection tests mentioned in the main text.
We still have to compute θ, m_1, and m_2, the details of which are given below.

Algorithm 1 Finding the origin of the ellipse

1: Scale the axis to transform the ellipsoid to a sphere.
2: Project the origin to the plane containing the triangle to find center of the circle (O) formed by the intersection of sphere with plane.
3: Rescale the axis back.

1.1 Computation of θ

We have

$$t^2\langle T,T\rangle_D + u^2\langle U,U\rangle_D + 2tu\langle T,U\rangle_D + \langle O,O\rangle_D - 1 = 0$$
$$t^2(m_2^2\cos^2\theta + m_1^2\sin^2\theta) + u^2(m_2^2\sin^2\theta + m_1^2\cos^2\theta) + tu(m_1^2 - m_2^2)\sin 2\theta - m_1^2 m_2^2 = 0$$

Consider

$$\frac{\text{coefficient of } tu}{\text{coefficient of } t^2 - \text{coefficient of } u^2} \equiv \frac{2\langle T,U\rangle_D}{\langle U,U\rangle_D - \langle T,T\rangle_D} = \tan(2\theta)$$
$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2\langle T,U\rangle_D}{\langle U,U\rangle_D - \langle T,T\rangle_D}\right) \tag{1}$$

1.2 Computation of m_1 and m_2

Consider

$$\frac{\text{coefficient of } t^2 + \text{coefficient of } u^2}{\text{coefficient of constant}} \equiv \frac{1}{m_1^2} + \frac{1}{m_2^2} = \frac{\langle T,T\rangle_D + \langle U,U\rangle_D}{1 - \langle O,O\rangle_D}$$

$$\frac{(\text{coefficient of } tu)^2 + (\text{coefficient of } u - \text{coefficient of } t)^2}{\text{coefficient of constant}^2} \equiv$$
$$\left(\frac{1}{m_2^2} - \frac{1}{m_1^2}\right)^2 = \frac{4\langle T,U\rangle_D^2 + (\langle U,U\rangle_D - \langle T,T\rangle_D)^2}{(1 - \langle O,O\rangle_D)^2}$$

Solving above two equations, we get

$$m_1 = \sqrt{2\left[\frac{1 - \langle O,O\rangle_D}{(\langle T,T\rangle_D + \langle U,U\rangle_D) - \sqrt{4\langle T,U\rangle_D^2 + (\langle T,T\rangle_D - \langle U,U\rangle_D)^2}}\right]}$$
$$m_2 = \sqrt{2\left[\frac{1 - \langle O,O\rangle_D}{(\langle T,T\rangle_D + \langle U,U\rangle_D) + \sqrt{4\langle T,U\rangle_D^2 + (\langle T,T\rangle_D - \langle U,U\rangle_D)^2}}\right]} \tag{2}$$

2 Circle-line intersection

Let $P_1(x_1, y_1)$ and $P_1(x_2, y_2)$ be the points and r be the radius of the circle. Consider the point $\alpha P_1 + (1-\alpha)P_2$, where the circle intersects the line joining P_1 and P_2. We have

$$(\alpha x_1 + (1-\alpha)x_1)^2 + (\alpha y_1 + (1-\alpha)y_1)^2 = r^2$$
$$\alpha^2((x_1 - x_2)^2 + (y_1 - y_2)^2) + 2\alpha(x_1x_2 - x_2^2 + y_1y_2 - y_2^2) + x_2^2 + y_2^2 - r^2 = 0.$$

Solving the above equation will give two values for α and the ones that satisfy $0 <= \alpha <= 1$ are the points of intersection of circle and the line

3 Jacobian of an ellipse

A point on the ellipse in 3D co-ordinates is given by $P = O + T_N m_1 \cos\phi + U_N m_2 \sin\phi$, where $T_N = T\cos\theta - U\sin\theta$, and $U_N = T\sin\theta + U\cos\theta$

The component of P along T_N-axis is $T_N' P$ and along U-axis is $U_N' P$ up to a translation determined by any static origin of the $T - U$ plane. Hence,

$$\frac{d(T_N'P)}{d\tau} = T_N'\frac{dO}{d\tau} + O'\frac{dT_N}{d\tau} + \frac{dm_1}{d\tau}\cos\phi; \frac{d(T_N'P)}{d\phi} = -m_1\sin\phi$$
$$\frac{d(U_N'P)}{d\tau} = U_N'\frac{dO}{d\tau} + O'\frac{dU_N}{d\tau} + \frac{dm_2}{d\tau}\sin\phi; \frac{d(U_N'P)}{d\phi} = \ m_2\cos\phi$$

The Jacobian in $T_N - U_N$ co-ordinate system is $d(T_N'P)d(U_N'P)$ and in $\tau - \phi$ coordinate system, the Jacobian is given by

$$\Bigg(m_2\frac{dm_1}{d\tau}\cos^2\phi + m_1\frac{dm_2}{d\tau}\sin^2\phi + (m_2\cos\phi T_N' + m_1\sin\phi U_N')\frac{dO}{d\tau}$$
$$+O'\left(\frac{dT_N}{d\tau}m_2\cos\phi + \frac{dU_N}{d\tau}m_1\sin\phi\right)\Bigg)d\tau d\phi$$

3.1 Derivation of $\frac{dT_n}{d\tau}$ and $\frac{dU_n}{d\tau}$

$$\frac{dT_N}{d\tau} = (-T\sin\theta - U\cos\theta)\frac{d\theta}{d\tau}$$
$$\frac{dU_N}{d\tau} = (T\cos\theta - U\sin\theta)\frac{d\theta}{d\tau}$$

From Equation 1, $\sin\theta = \frac{2\langle T,U\rangle_D}{\Delta}$ and $\cos\theta = \frac{\langle U,U\rangle_D - \langle T,T\rangle_D}{\Delta}$, where $\Delta = \sqrt{4\langle T,U\rangle_D^2 + (\langle T,T\rangle_D - \langle U,U\rangle_D)^2}$

$$\cos\theta\frac{d\theta}{d\tau} = 2\frac{\Delta\langle T,U\rangle_E - \langle T,U\rangle_D\frac{d\Delta}{d\tau}}{\Delta^2}$$
$$-\sin\theta\frac{d\theta}{d\tau} = \frac{\Delta(\langle U,U\rangle_E - \langle T,T\rangle_E) - (\langle U,U\rangle_D - \langle T,T\rangle_D)\frac{d\Delta}{d\tau}}{\Delta^2}$$
$$\frac{d\Delta}{d\tau} = \frac{1}{\Delta}(4\langle T,U\rangle_D\langle T,U\rangle_E + (\langle T,T\rangle_E - \langle U,U\rangle_E)(\langle T,T\rangle_D - \langle U,U\rangle_D))\Big)$$
$$\text{where, } E = \begin{bmatrix} \frac{-1}{a^3} & 0 & 0 \\ 0 & \frac{-a}{b^4} & 0 \\ 0 & 0 & \frac{-a}{b^4} \end{bmatrix}$$

3.2 Derivation of $\dfrac{dm_1}{d\tau}$ and $\dfrac{dm_2}{d\tau}$

From 2, we have

$$\begin{aligned} m_1^2 &= \frac{NR}{DR_1}, \text{where} \\ NR &= 2(1 - \langle O, O \rangle_D); DR_1 = \langle T, T \rangle_D + \langle U, U \rangle_D - \Delta \\ \mathrm{d}NR &= -2\langle O, O \rangle_E d\tau - 4\langle O, dO \rangle_D \mathrm{d}DR_1 = \langle T, T \rangle_E + \langle U, U \rangle_E - d\Delta \text{ and} \\ m_2 \frac{dm_1}{d\tau} &= \frac{m_2}{m_1} \frac{-DR_1(\langle O, O \rangle_E + 2\langle O, \frac{dO}{d\tau} \rangle_D) - (1 - \langle O, O \rangle_D)\mathrm{d}DR_1}{DR_1^2} \end{aligned}$$

Also, from 2, we have

$$\begin{aligned} m_2^2 &= \frac{NR}{DR_2}, \text{where } DR_2 = \langle T, T \rangle_D + \langle U, U \rangle_D + \Delta \\ \mathrm{d}DR_2 &= \langle T, T \rangle_E + \langle U, U \rangle_E + d\Delta \\ m_1 \frac{dm_2}{d\tau} &= \frac{m_1}{m_2} \frac{-DR_2(\langle O, O \rangle_E + 2\langle O, \frac{dO}{d\tau} \rangle_D) - (1 - \langle O, O \rangle_D)\mathrm{d}DR_2}{DR_2^2} \end{aligned}$$

3.3 Derivation of $\dfrac{dO}{d\tau}$

From Section 1,

$$\begin{aligned} \langle T, O \rangle_D &= 0 \\ \langle U, O \rangle_D &= 0 \\ \hat{n}^T(O - C_1) &= 0 \end{aligned}$$

Let $\alpha_1 = a$, $\alpha_2 = b$, $\alpha_3 = b$. Hence, $\alpha_i d\alpha_i = \tau d\tau/4$; $i = \{1, 2, 3\}$
$\langle T, O \rangle_D = 0 \Rightarrow \sum_i \frac{t_i O_i}{\alpha_i^2} = 0$
Taking derivative with respect to τ on both sides,
we have $\sum_i \frac{t_i}{\alpha_i^2} \frac{dO_i}{d\tau} = \frac{\tau}{2} \sum_i \frac{t_i O_i}{\alpha_i^4}$
Similarly, $\sum_i \frac{u_i}{\alpha_i^2} \frac{dO_i}{d\tau} = \frac{\tau}{2} \sum_i \frac{u_i O_i}{\alpha_i^4}$, and $\sum_i n_i O_i = 0$
Using the above three equations, we have

$$\frac{dO}{d\tau} = \frac{\tau}{2} \begin{bmatrix} \frac{t_1}{a^2} & \frac{t_2}{b^2} & \frac{t_3}{b^2} \\ \frac{u_1}{a^2} & \frac{u_2}{b^2} & \frac{u_3}{b^2} \\ n_1 & n_2 & n_3 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i \frac{t_i O_i}{\alpha_i^4} \\ \sum_i \frac{u_i O_i}{\alpha_i^4} \\ 0 \end{bmatrix}$$

4 Transient renderer

$$I = \int_{\bar{x} \in \Omega} f(\bar{x}) d\mu(\bar{x}), \text{where } \Omega \text{ is set of all paths}$$

The transient rendering is defined as

$$
\begin{aligned}
I(t) &= \int_{\Omega} f(\bar{x})\delta(|\bar{x}| - t)d\mu(\bar{x}), \text{where } |\bar{x}| \text{ denotes the length of the path} \\
&= \int_{\Omega_1}\int_{x\in\mathcal{M}}\int_{\Omega_2} f(\bar{x}_1)f(x_1 \to x \to x_2)f(\bar{x}_2)\delta(|\bar{x}| - t)d\mu(\bar{x}) \\
&= \int_{\Omega_1}\int_{\substack{x\in\mathcal{M}\,\cap \\ |\bar{x}_1|+|\bar{x}_2|+|x_1\to x\to x_2|=t}}\int_{\Omega_2} f(\bar{x}_1)f(x_1 \to x \to x_2)f(\bar{x}_2)d\mu(\bar{x})
\end{aligned}
$$

where x_1 denotes the end vertex of path $\bar{x}_1$, x_2 denotes the starting vertex of path $\bar{x}_2$, and x is an intermediate vertex. Let

$$
\begin{aligned}
|\bar{x}_1| &= t_1 \\
|\bar{x}_2| &= t_2 \\
t_1 + t_2 &< t \\
\tau &= t - (t_1 + t_2)
\end{aligned}
$$

Let us assume that the mesh ($\mathcal{M}$) is completely made of surface geometries (no volumetric scattering medium), then we have

$$
\begin{aligned}
I(t) &= \int_{\Omega_1}\int_{\substack{x\in\mathcal{M}\,\cap \\ |x_1\to x\to x_2|=\tau}}\int_{\Omega_2} f(\bar{x}_1)f(x_1 \to x \to x_2)f(\bar{x}_2)d\mu(\bar{x}_1)d\mu(\bar{x}_2)dA(x) \\
&= \int_{\Omega_1}\int_{\Omega_2} f(\bar{x}_1)f(\bar{x}_2)\int_{\substack{x\in\mathcal{M}\,\cap \\ |x_1\to x\to x_2|=\tau}} f(x_1 \to x \to x_2)dA(x)d\mu(\bar{x}_1)d\mu(\bar{x}_2)
\end{aligned}
$$

Let us assume that the mesh is represented as polygonal primitives. Let $\mathbf{T}$ be the set of all primitives which contains points that satisfy the temporal constraint $|x_1 \to x \to x_2| = \tau$. For one such primitive $\mathbf{T}r$ the points that satisfy the temporal constraint will lie on a ellipse (or partial ellipse). For these points, the transient intensity flow will become

$$
\begin{aligned}
I_{Tr}(t) = \int_{\Omega_1}\int_{\Omega_2} f(\bar{x}_1)f(\bar{x}_2)\int_{m_1}\int_{\phi} \frac{|\cos(x_1 \to x, \hat{n})|}{|x_1 \to x|^2}\frac{|\cos(x \to x_2, \hat{n})|}{|x_2 \to x|^2} \\
J d\tau d\phi d\mu(\bar{x}_1)d\mu(\bar{x}_2)
\end{aligned}
$$

where τ is the major axis of the ellipse and $\hat{n}$ is the normal of the primitive.

For the complete mesh,

$$
I(t) = \sum_{Tr} I_{Tr}(t)
$$