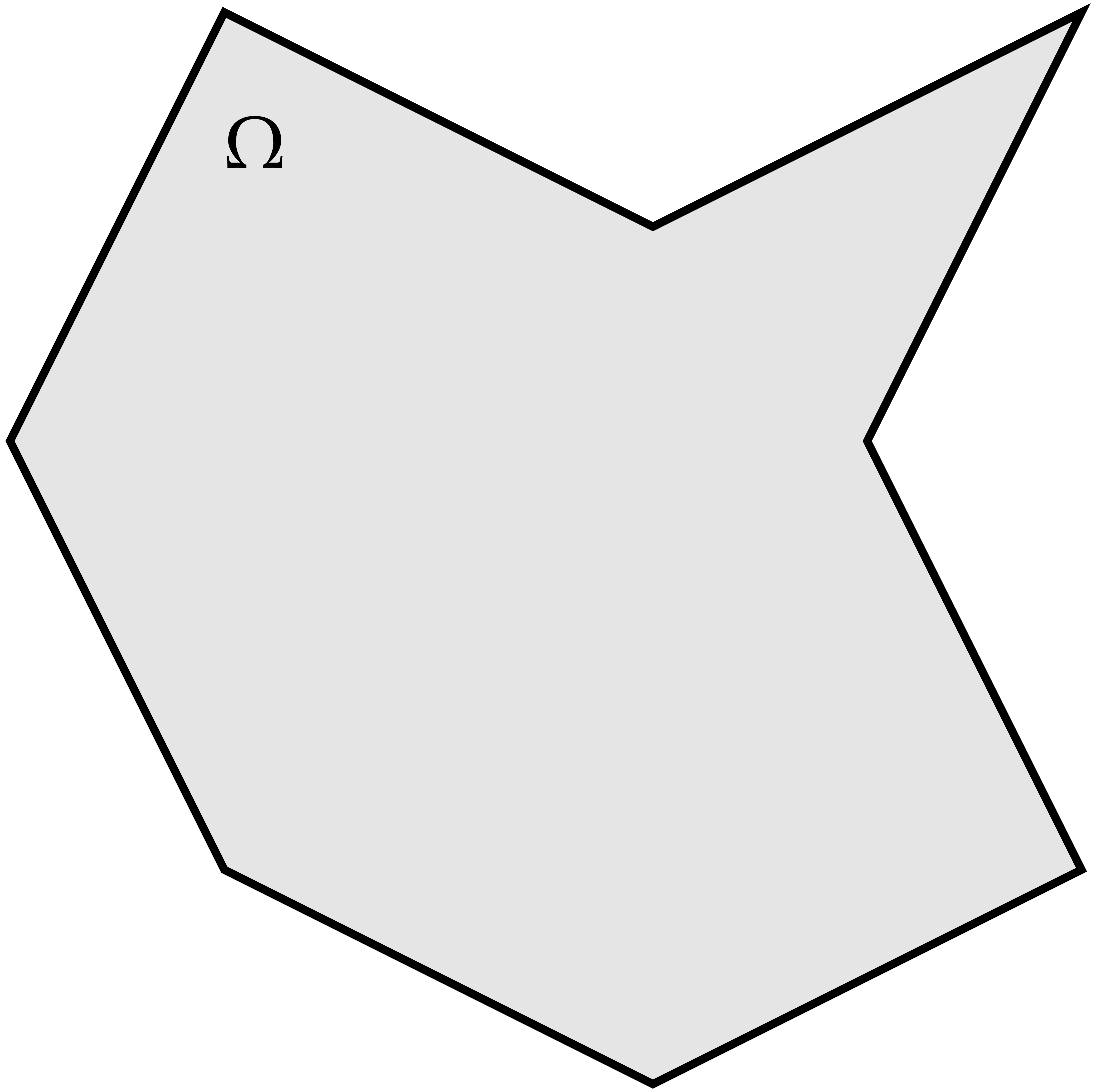
Boundary Value Caching for Walk on Spheres

Bailey Miller*, Rohan Sawhney*, Keenan Crane† , and Ioannis Gkioulekas†

Grid-Free Monte Carlo PDE Solvers

Walk on Spheres avoids **meshing** or **global solves**!



Poisson equation

$$\Delta u = f \quad \text{on } \Omega$$

$$u = g \quad \text{on } \partial\Omega_D$$

$$\frac{\partial u}{\partial n} = h \quad \text{on } \partial\Omega_N$$

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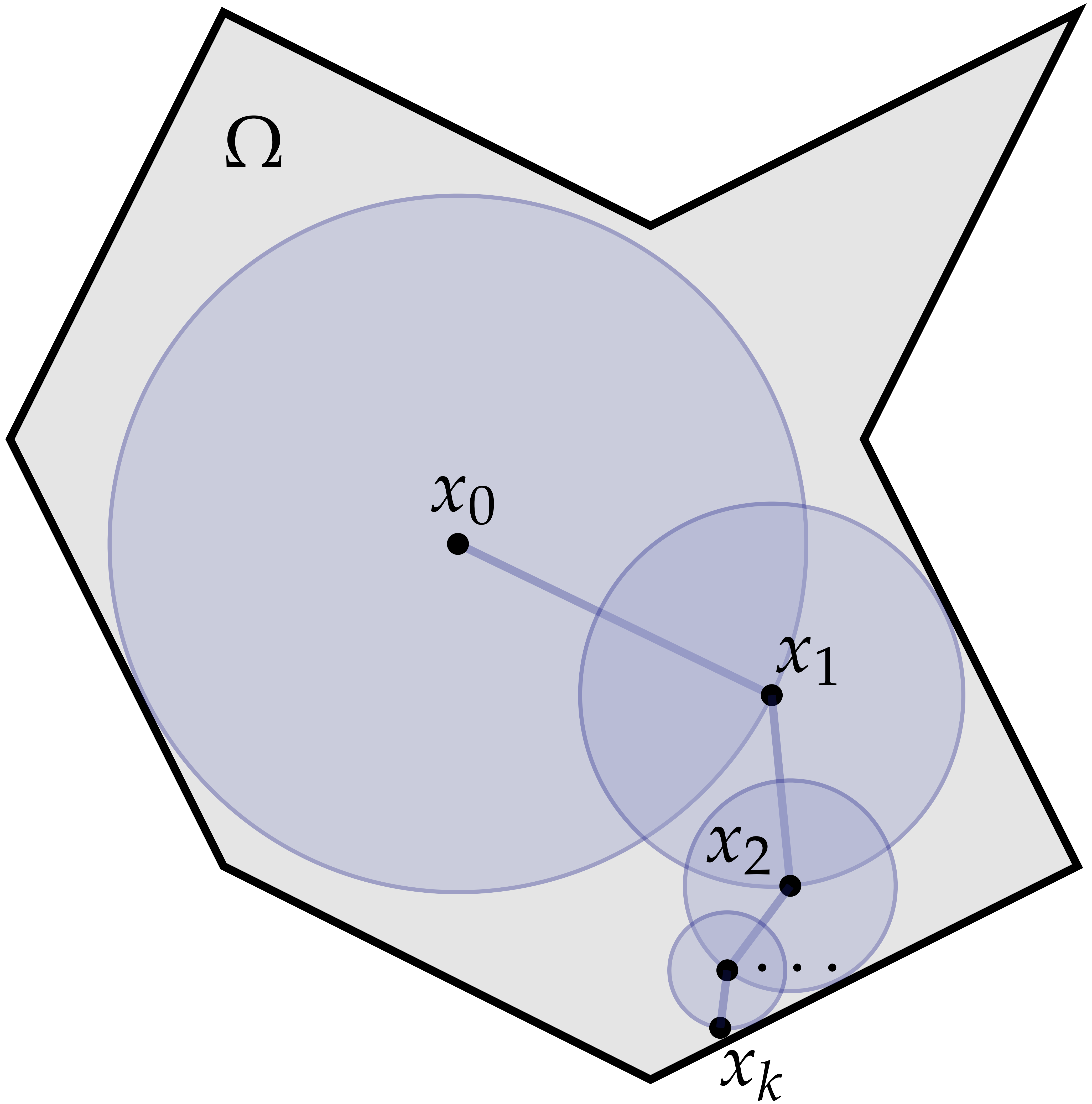

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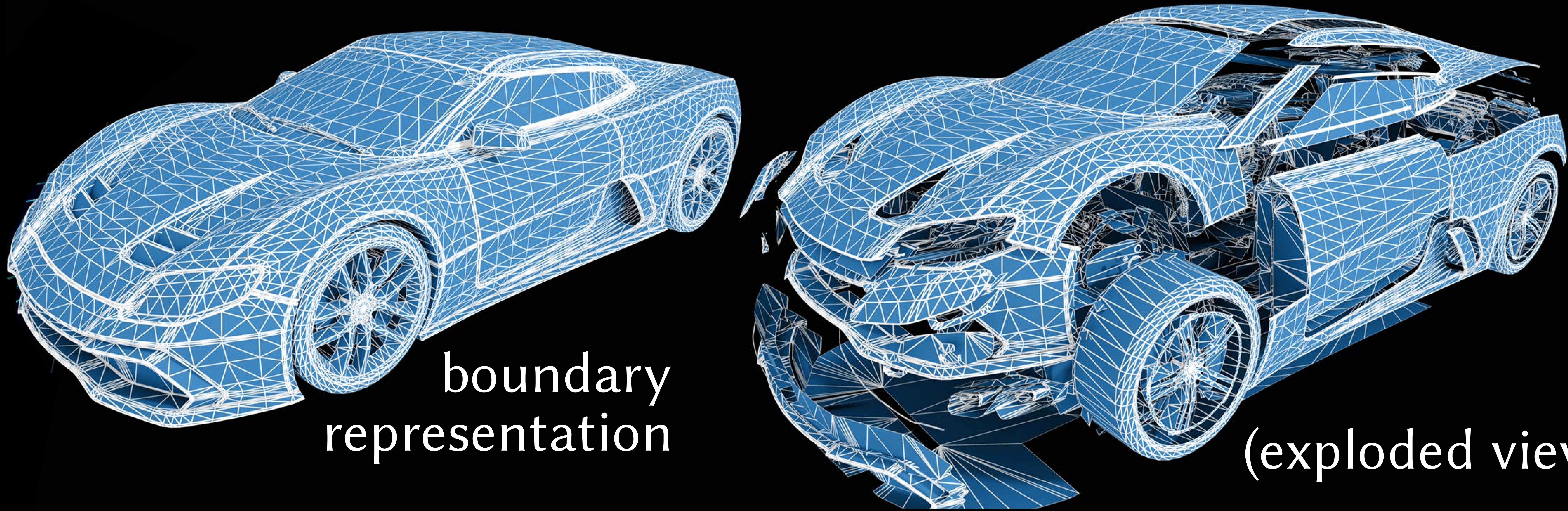
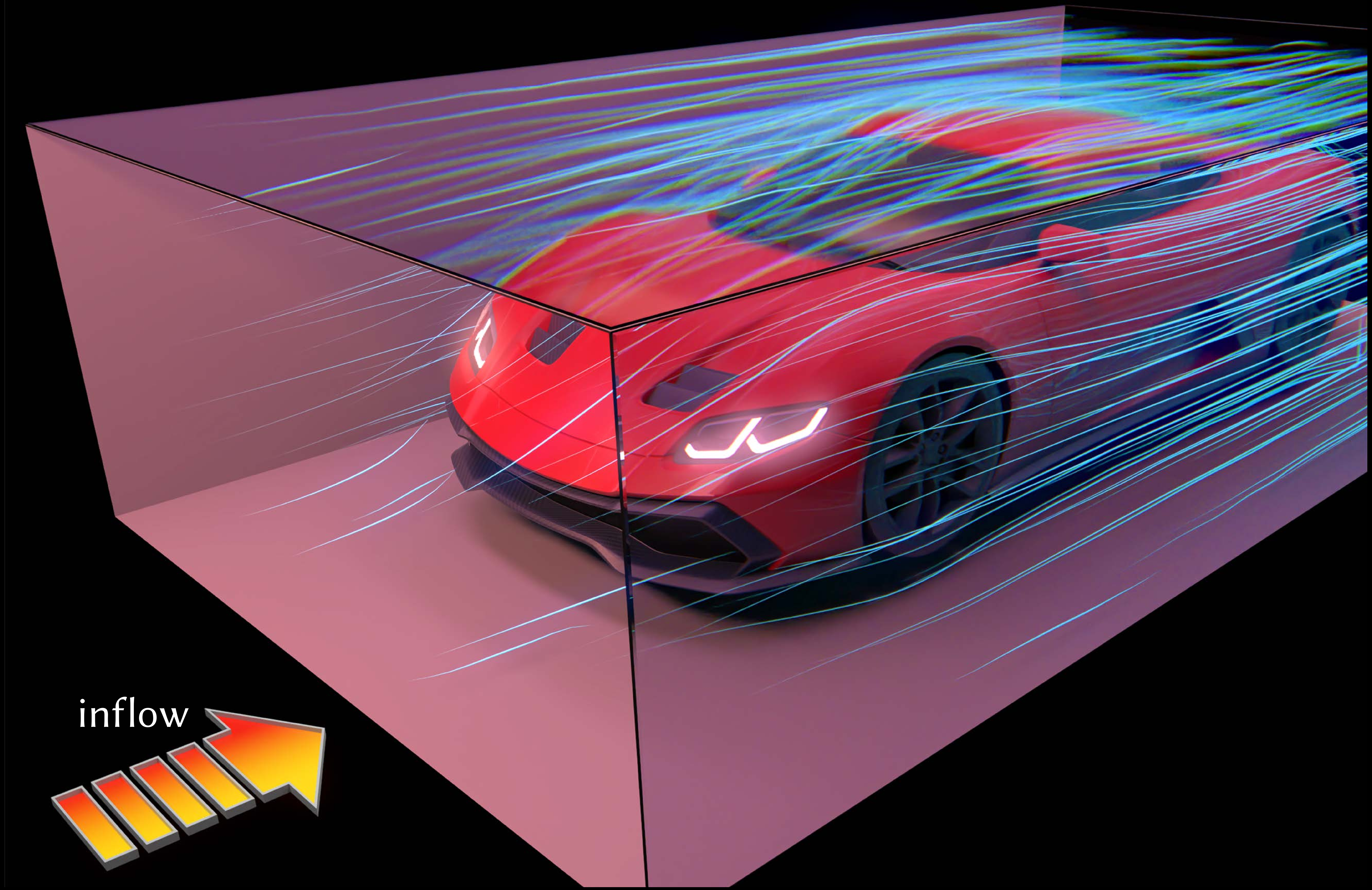
Robustly handle meshes intended for visualization



Potential Flow Simulation



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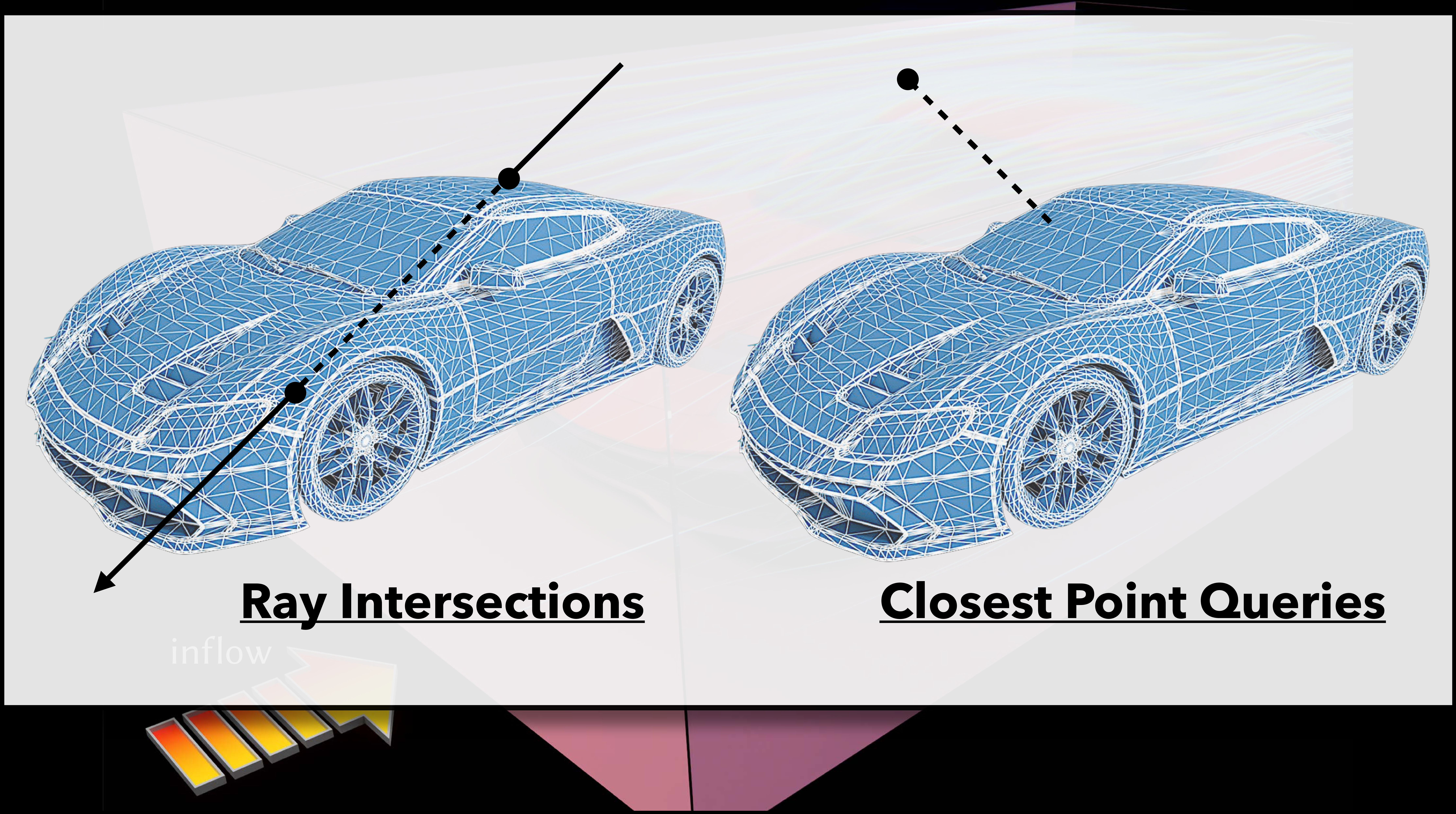
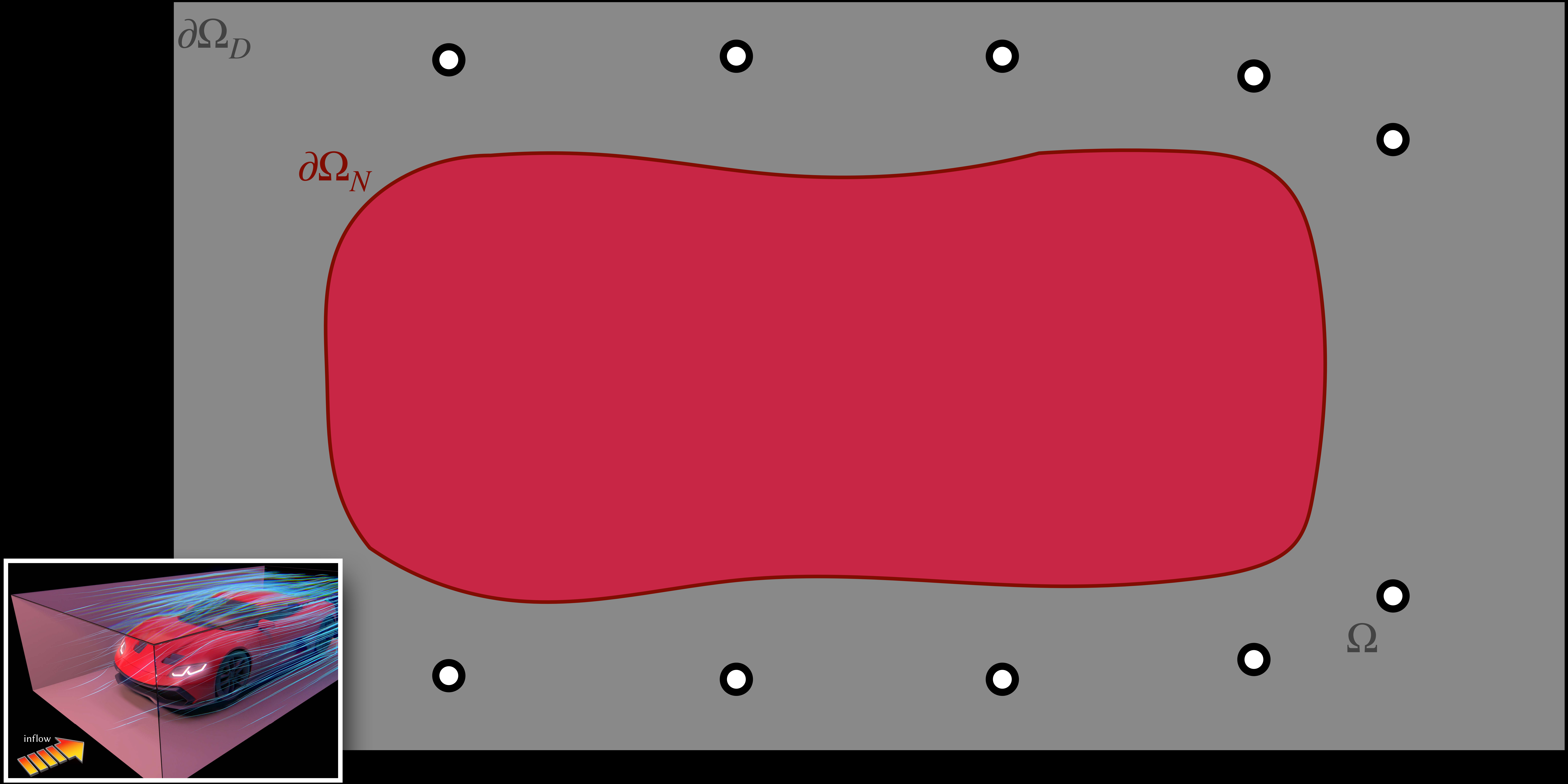
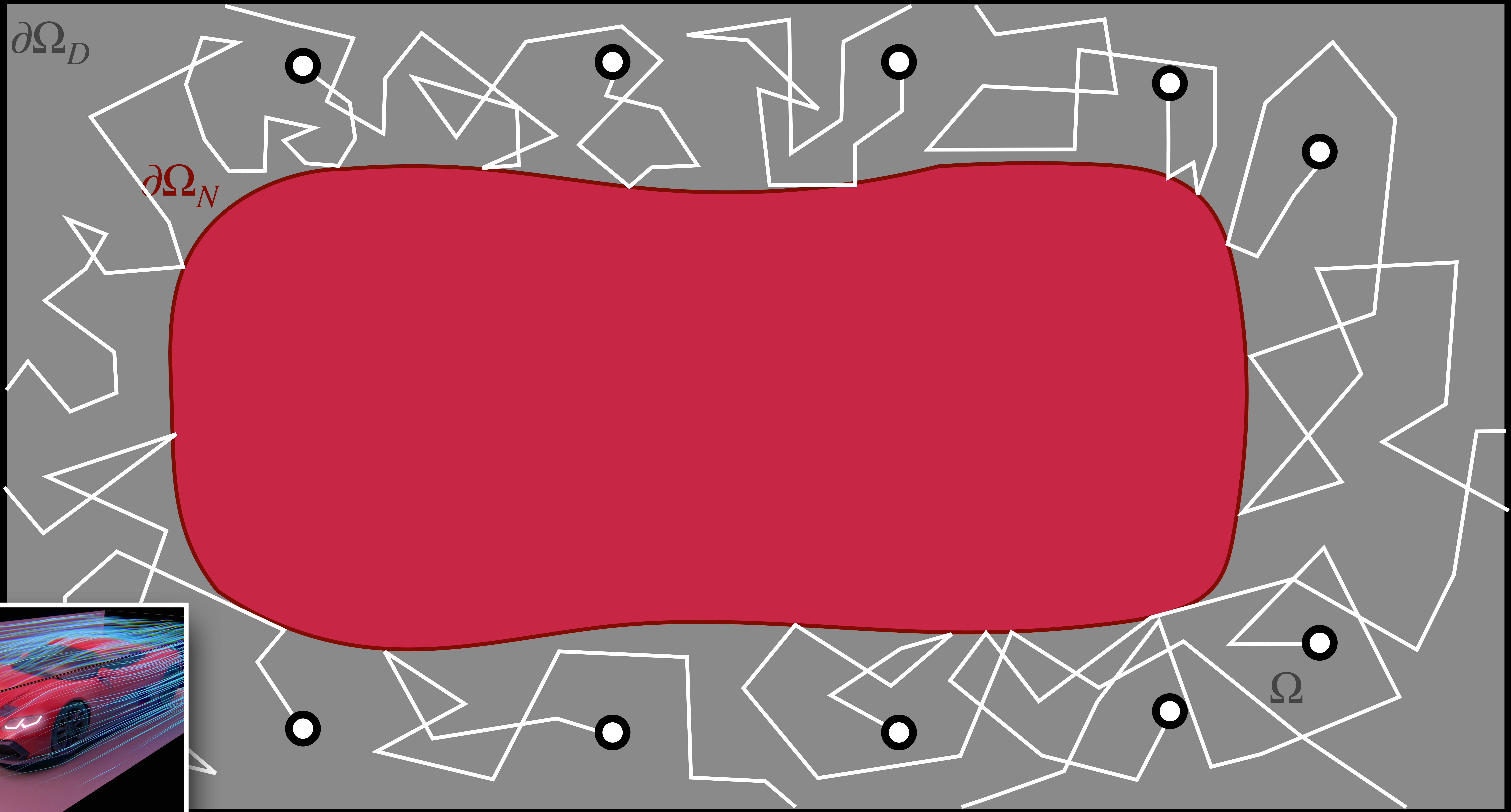
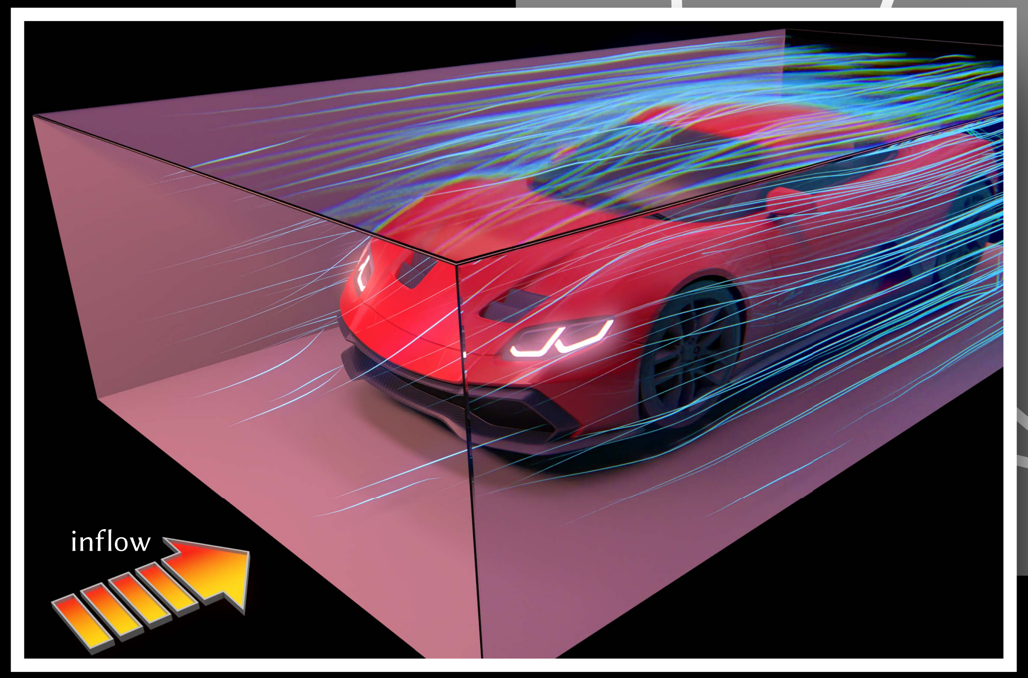




wind tunnel

pointwise estimator





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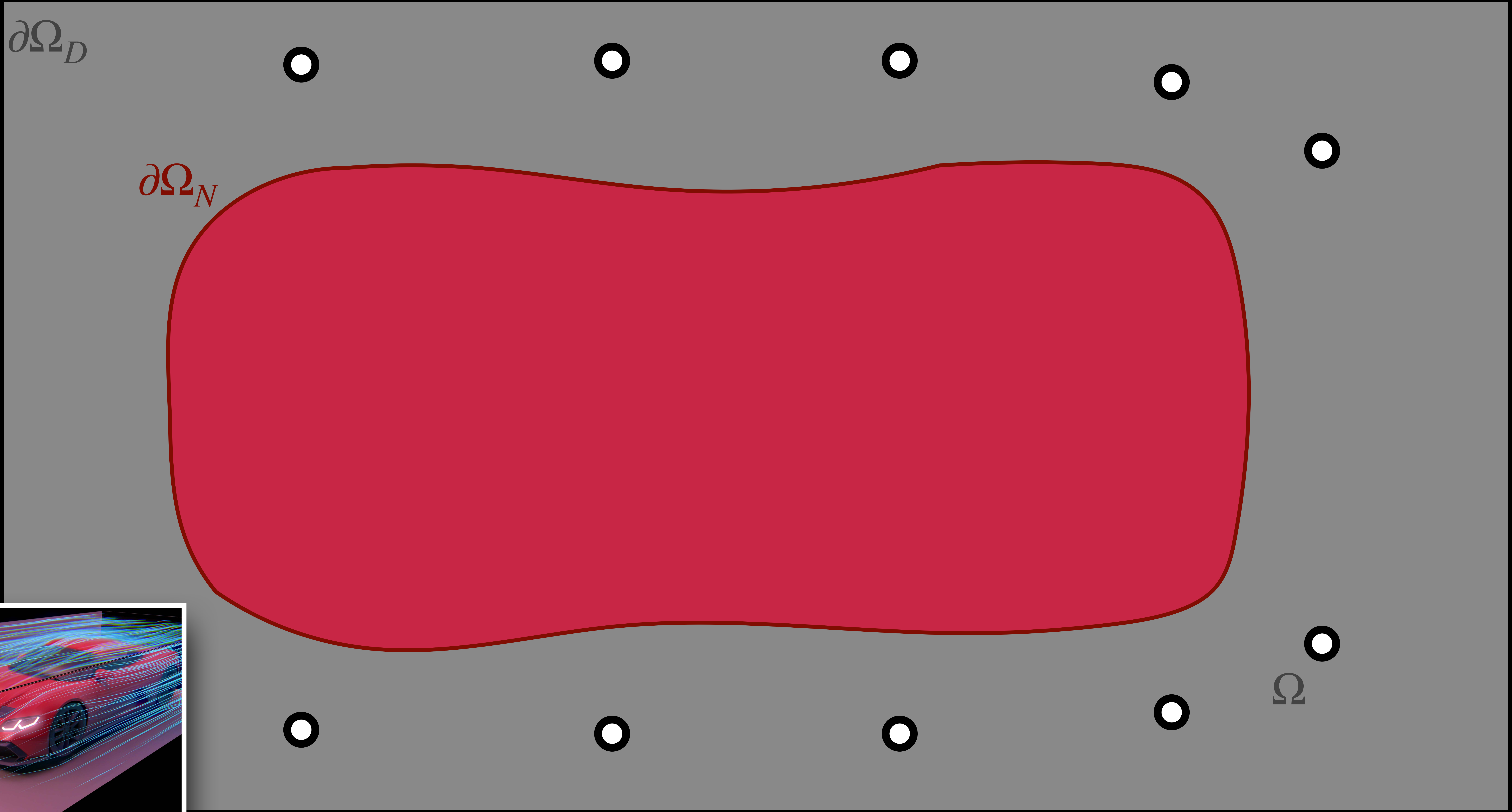

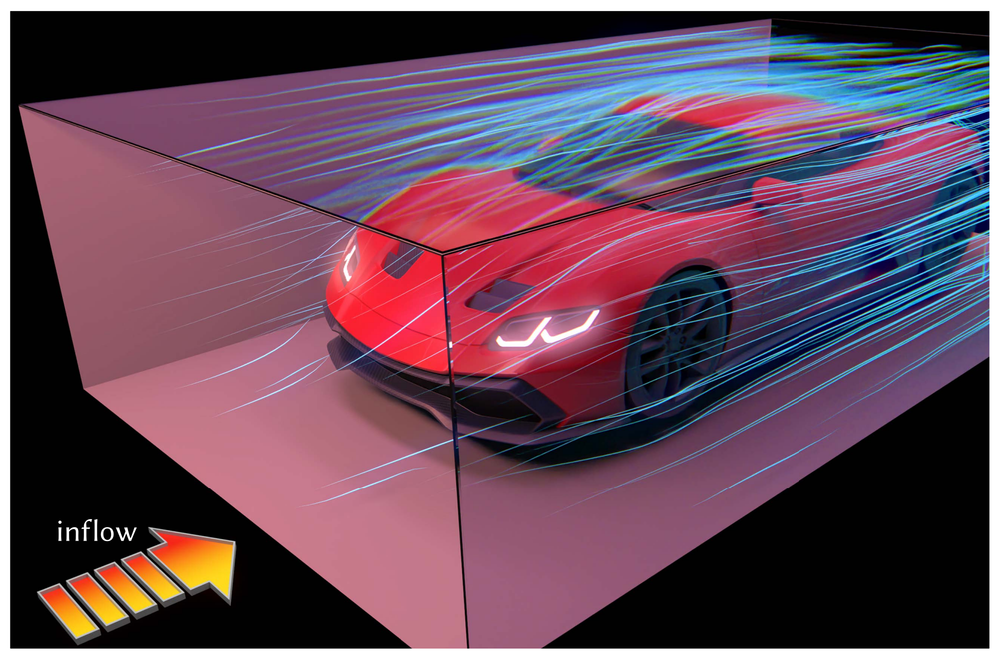

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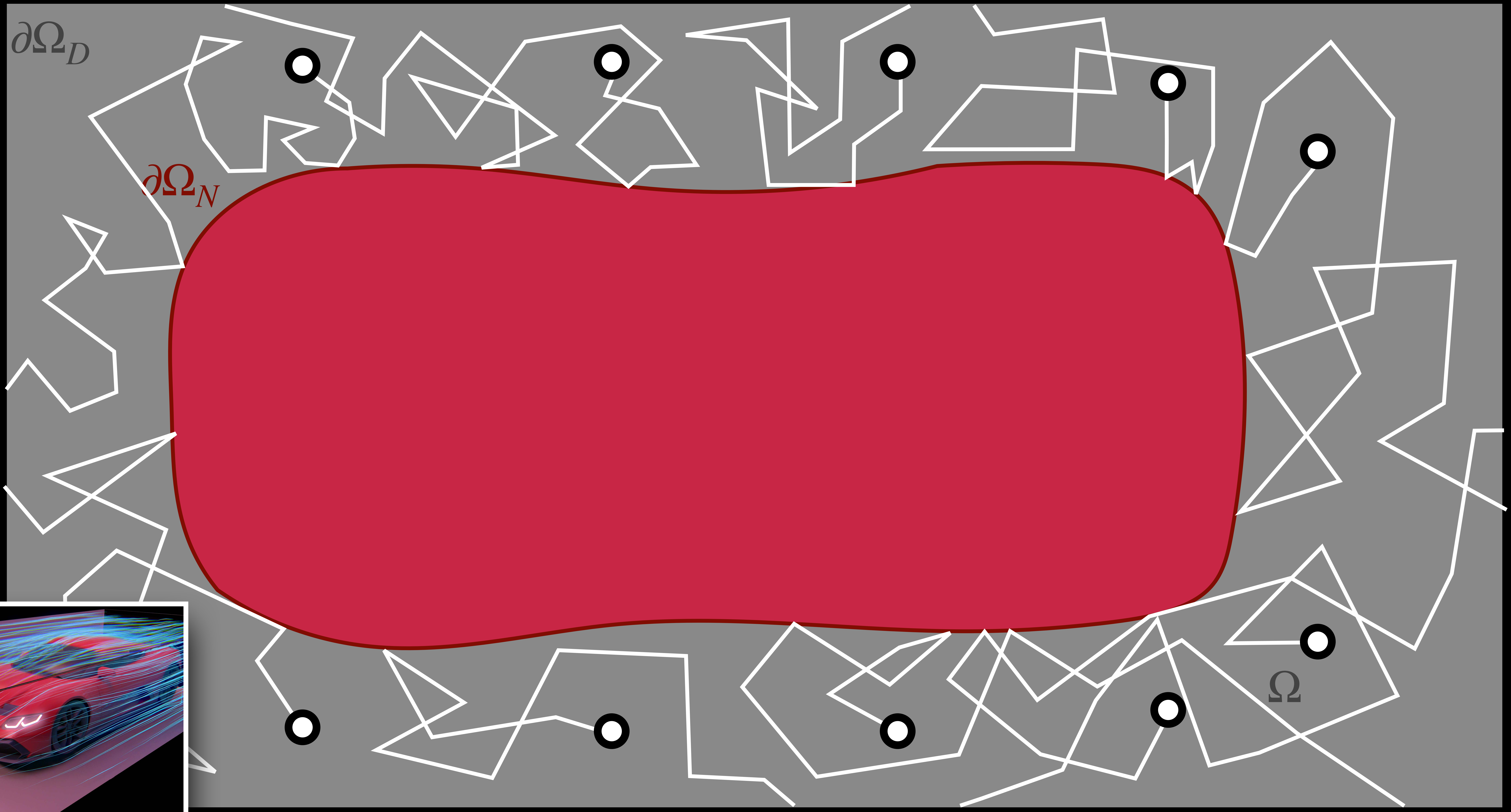

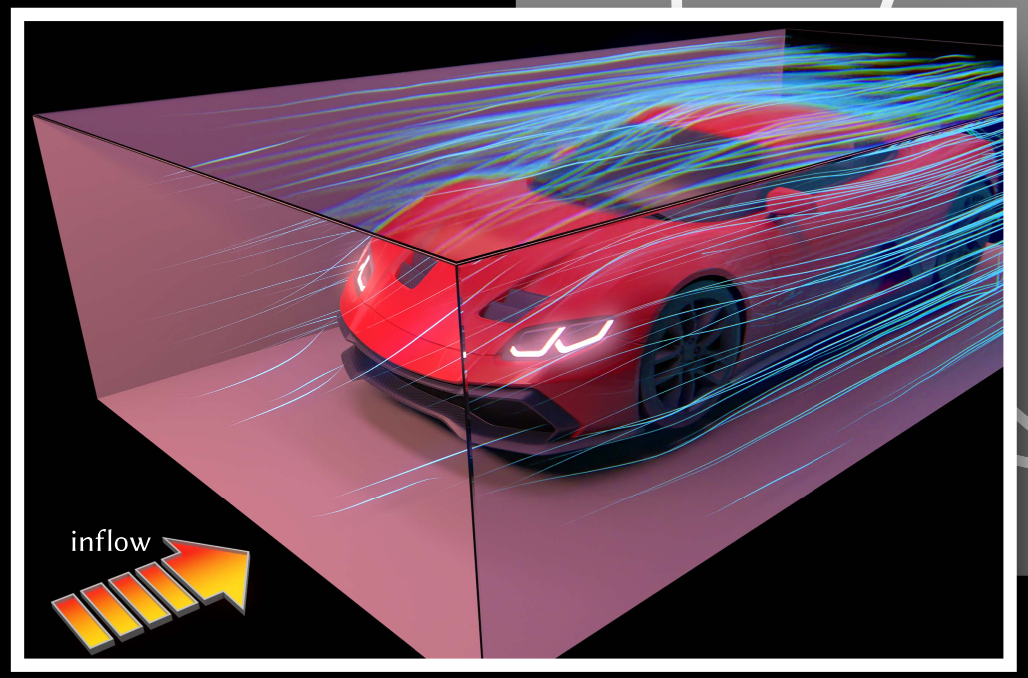

wind tunnel

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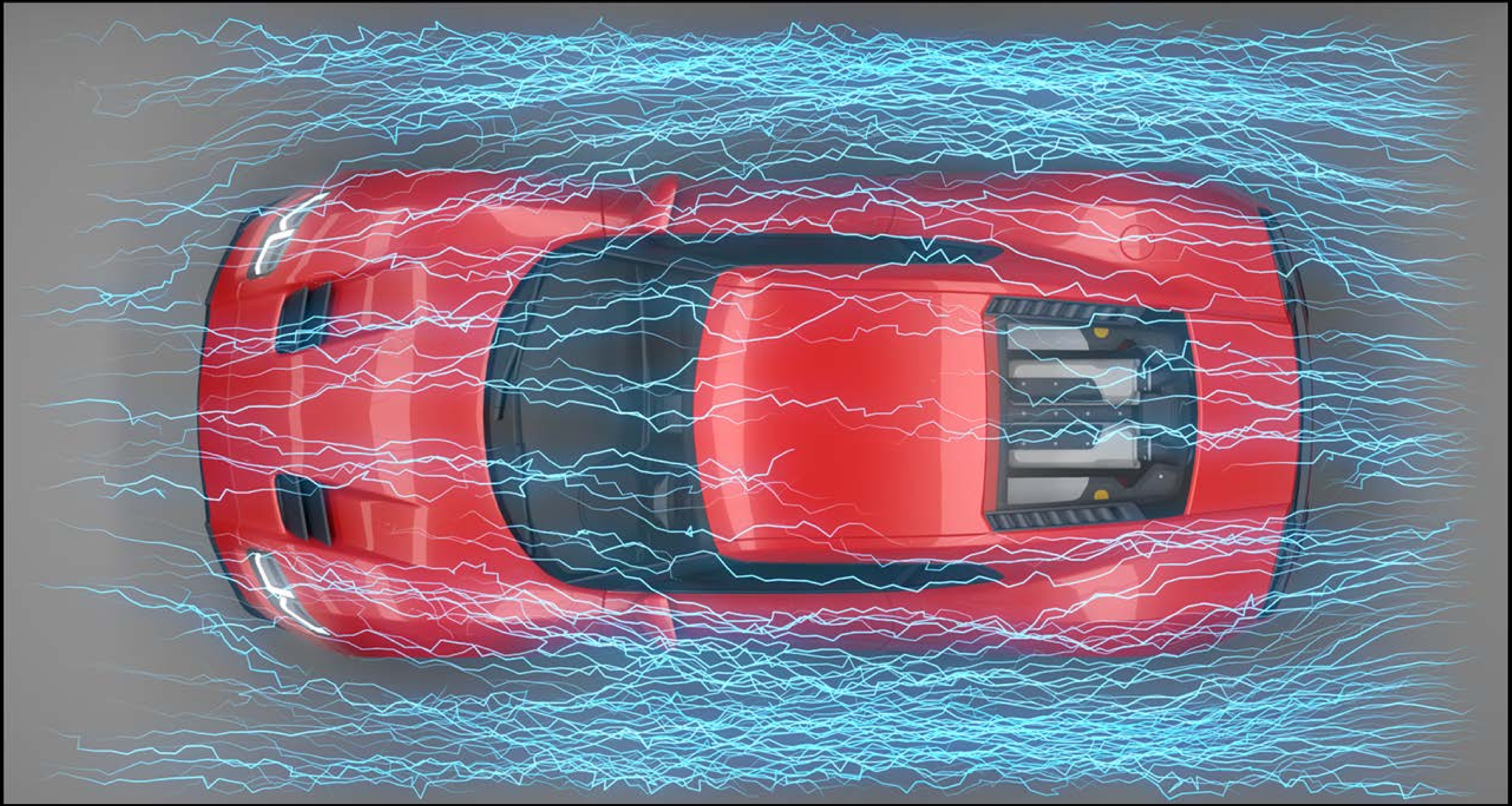
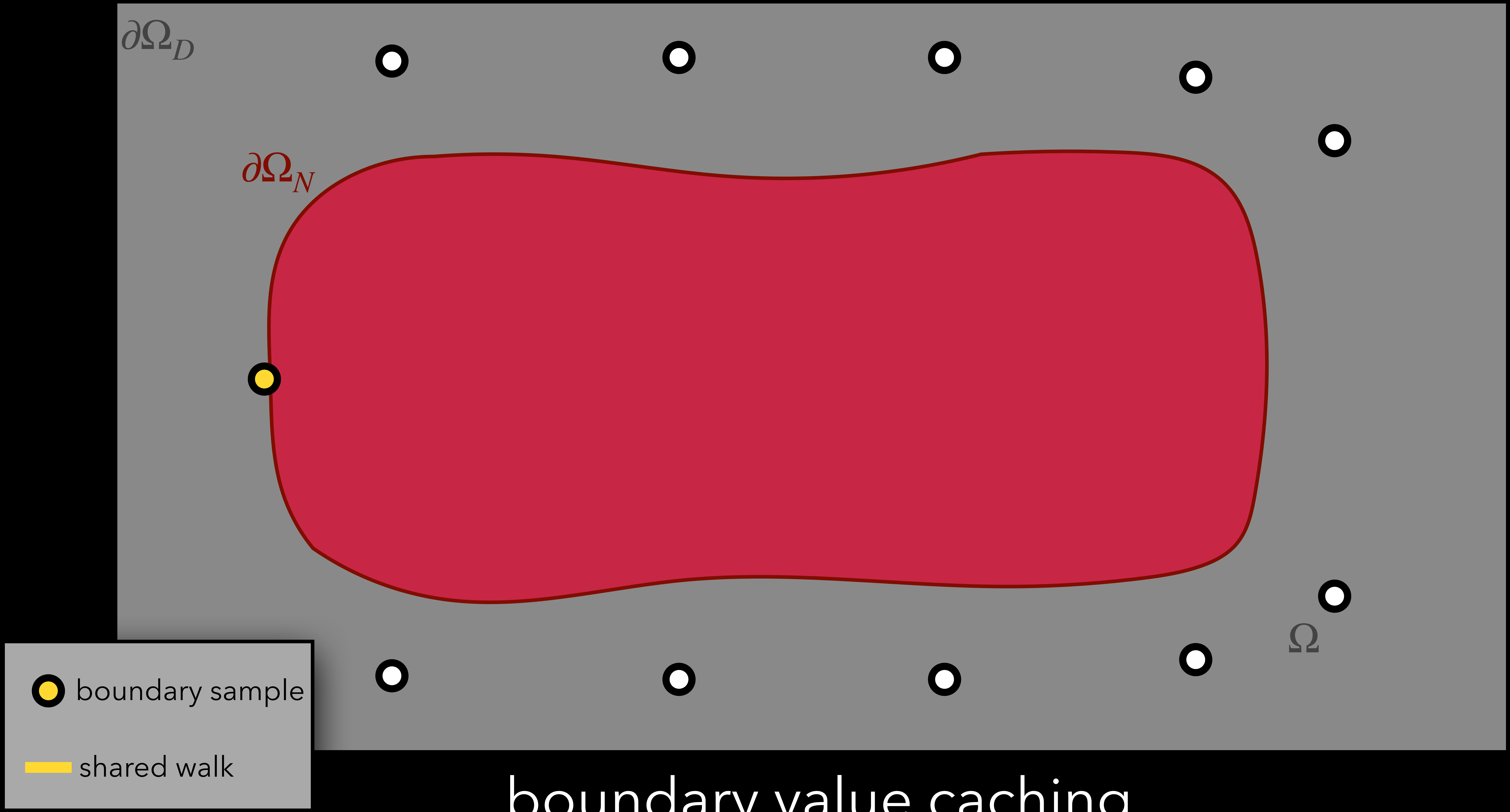
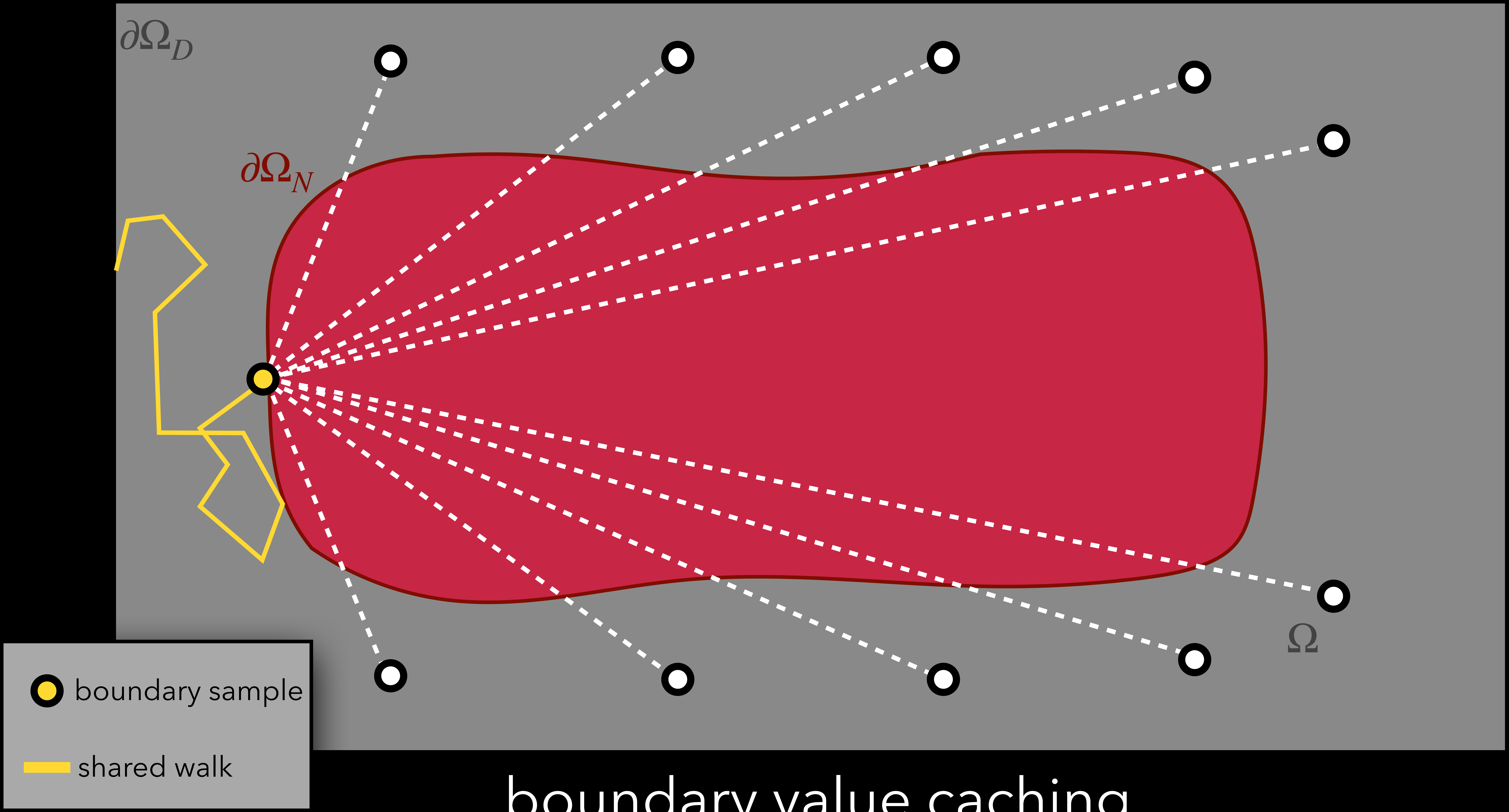
Noisy streamlines!



pointwise estimator



boundary value caching



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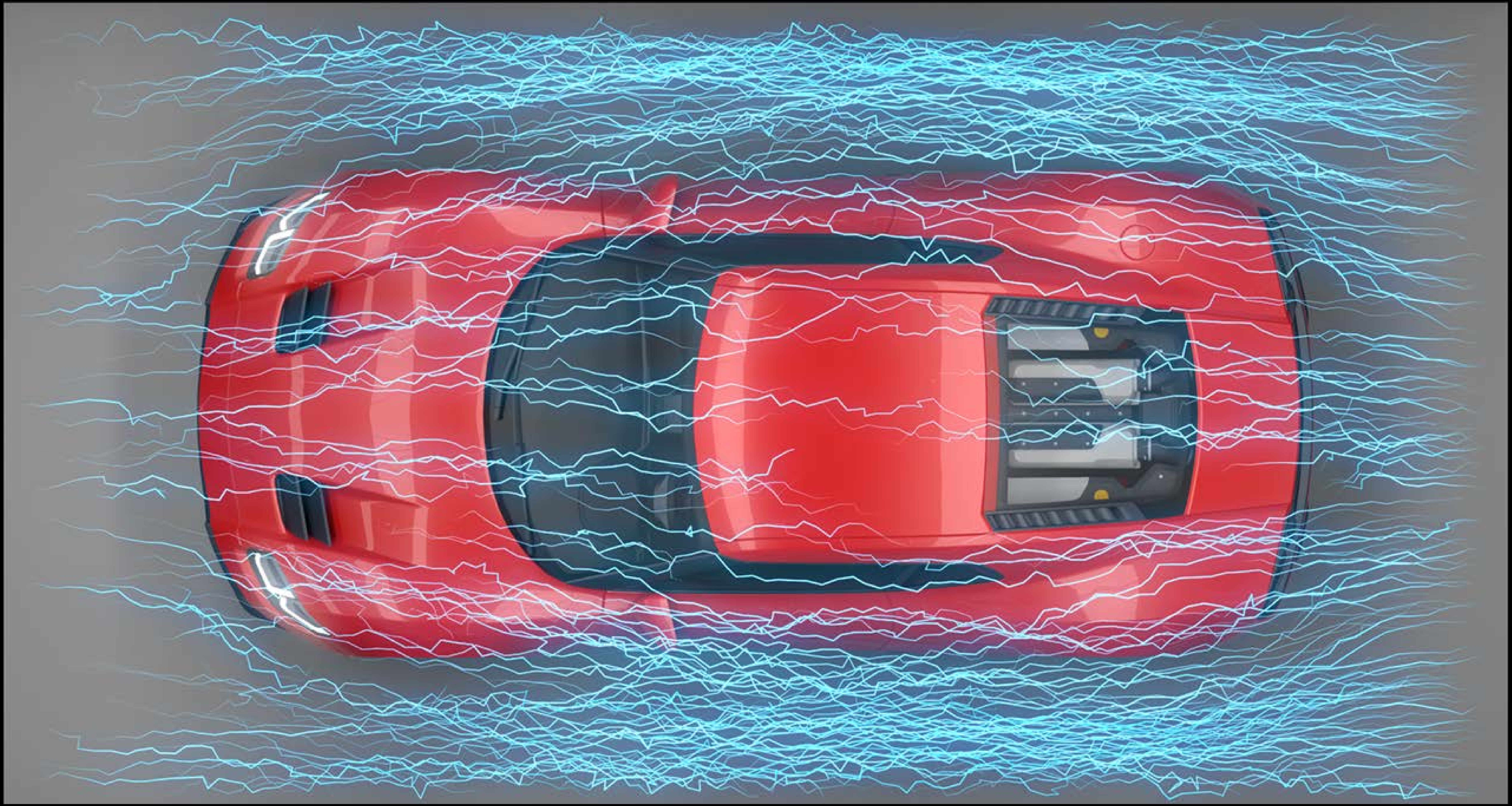

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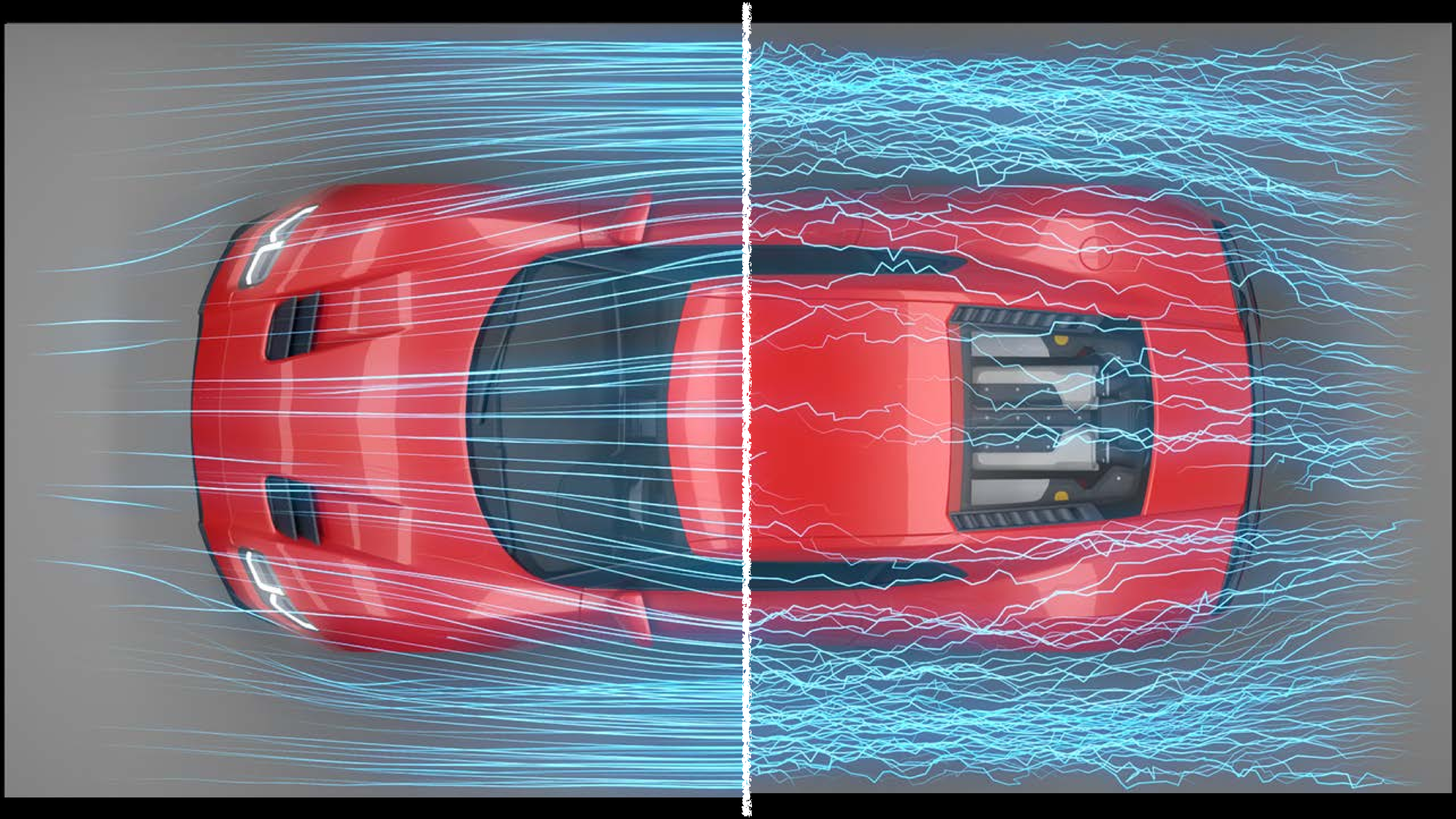


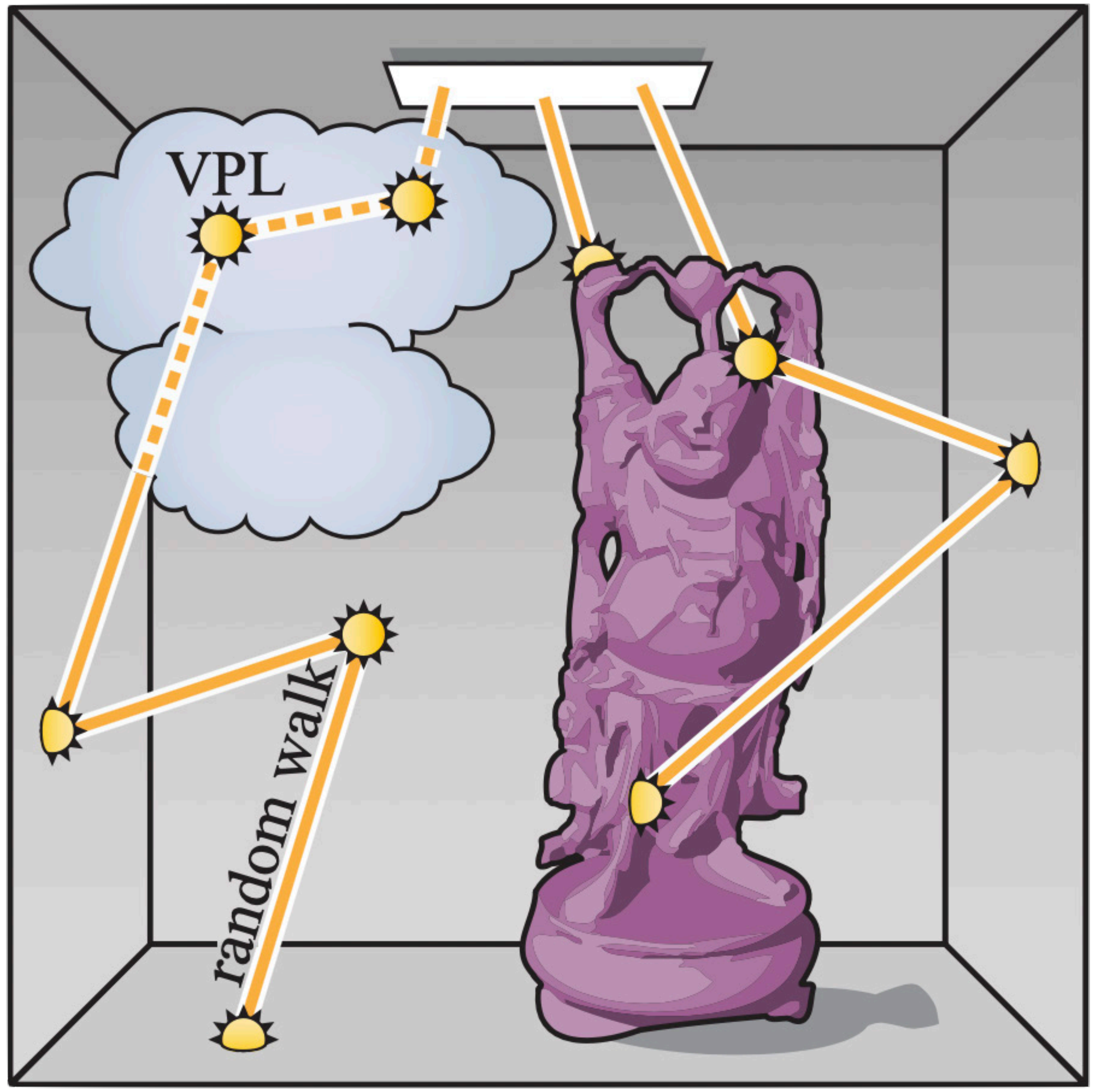
boundary value caching pointwise estimator

BACKGROUND

Sample Reuse in Monte Carlo Ray Tracing

Sample Reuse in Monte Carlo Ray Tracing

Virtual Point Light Methods (VPLs)



Sample Reuse in Monte Carlo Ray Tracing

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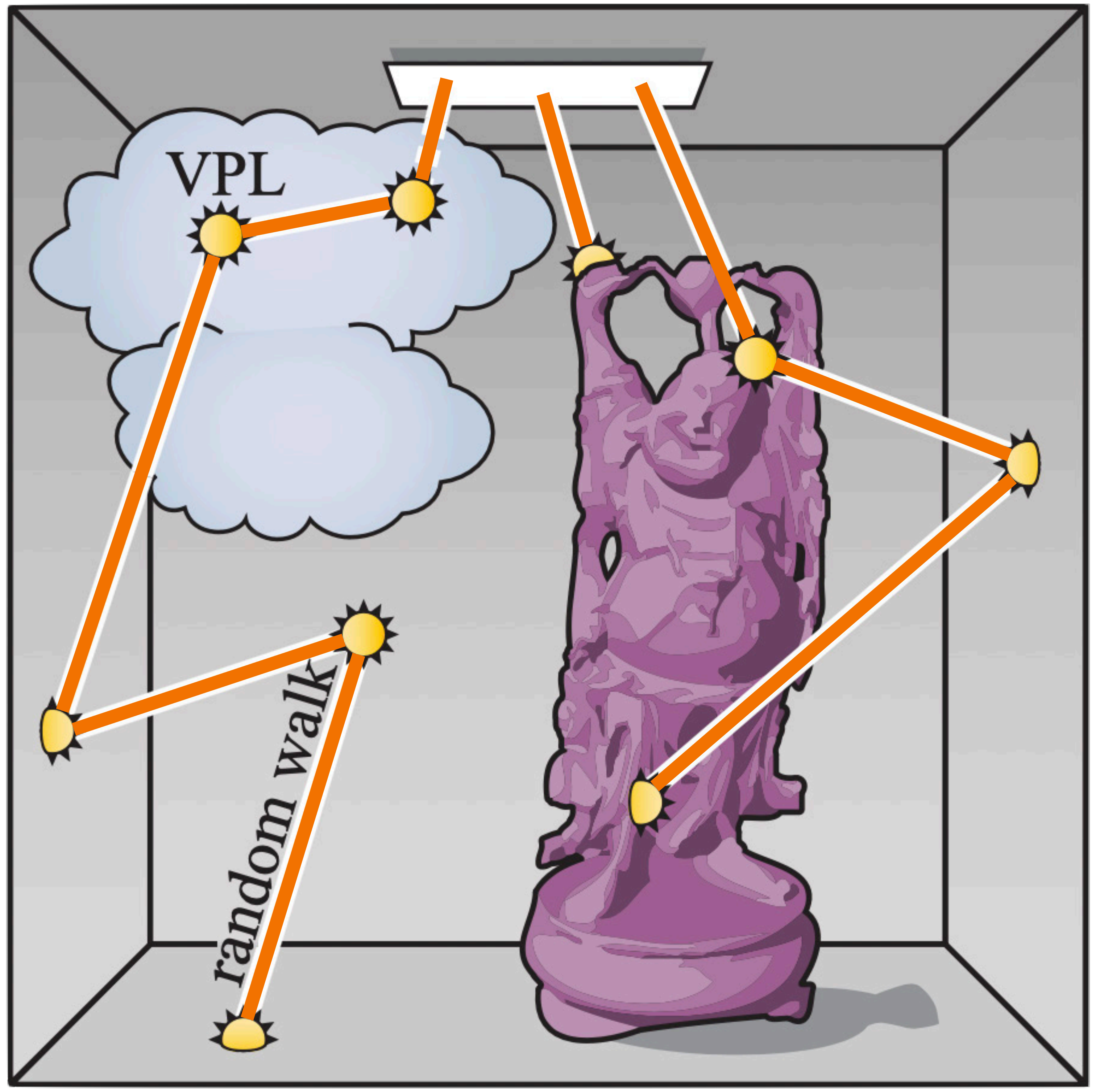

Step 1: Deposit radiance estimates

Sample Reuse in Monte Carlo Ray Tracing

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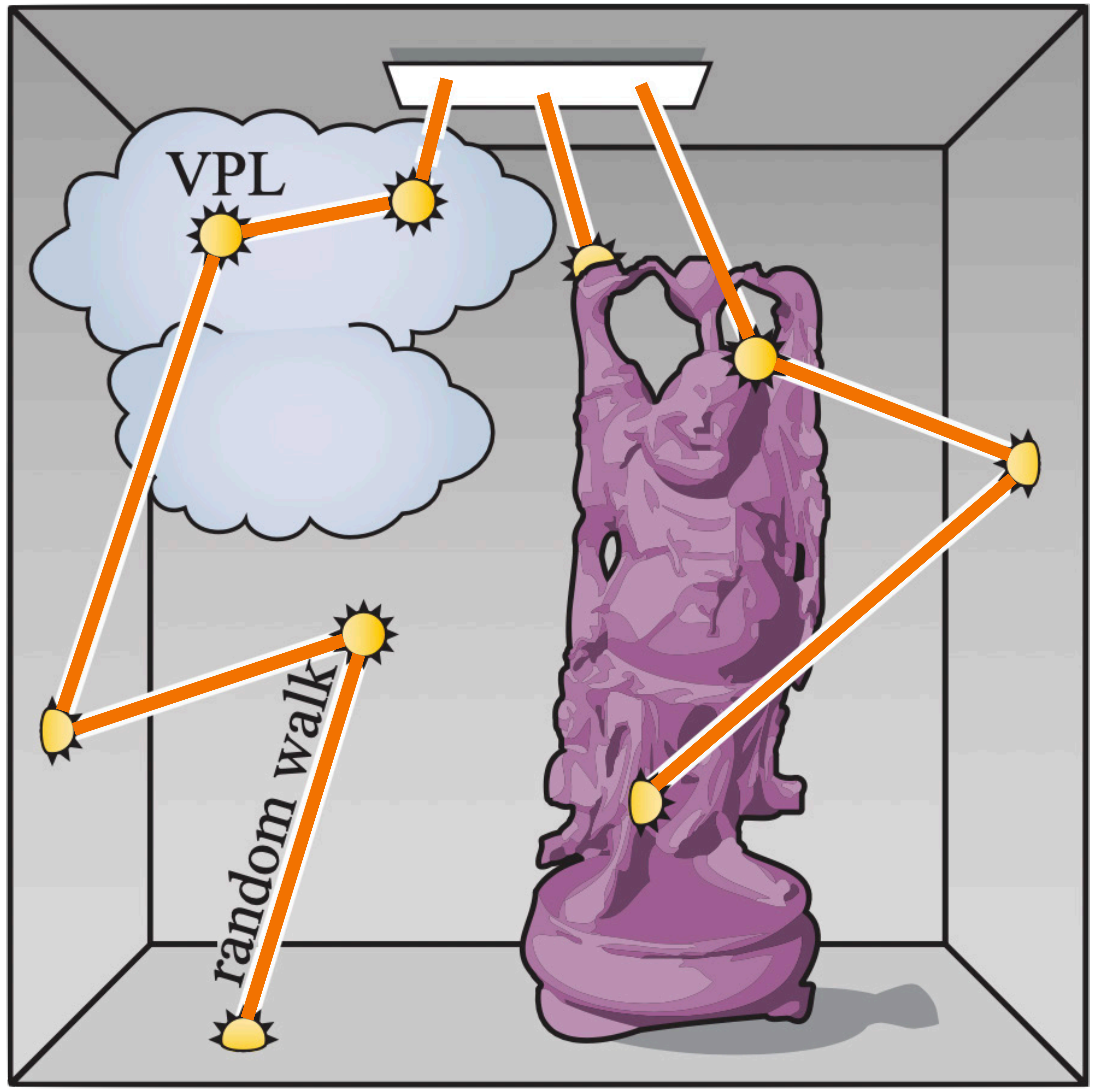

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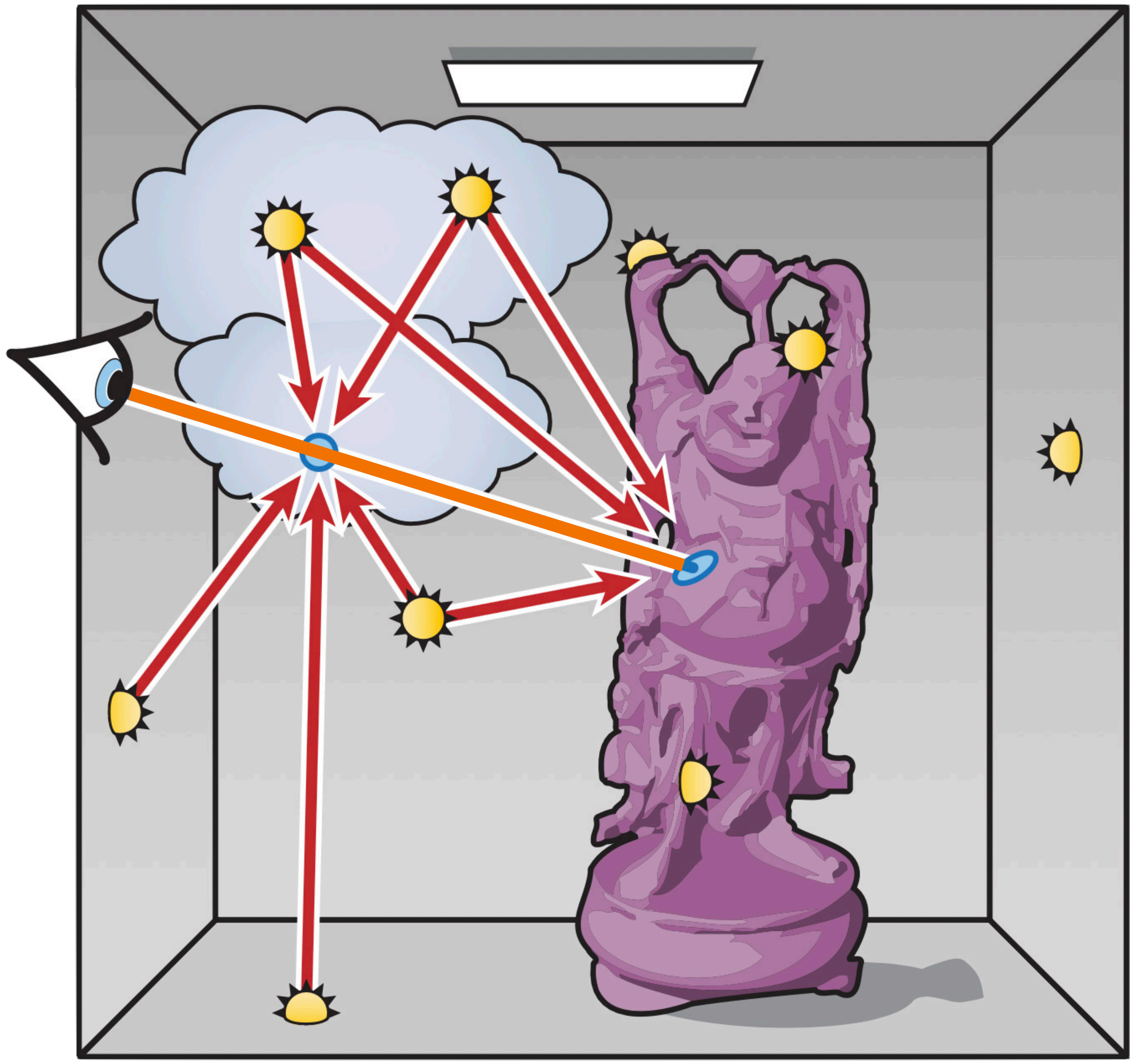

Step 2: Reuse cached radiance estimates

Boundary Integral Equation (BIE)

Laplace equation

$$\Delta u = 0 \quad \text{on } \Omega$$

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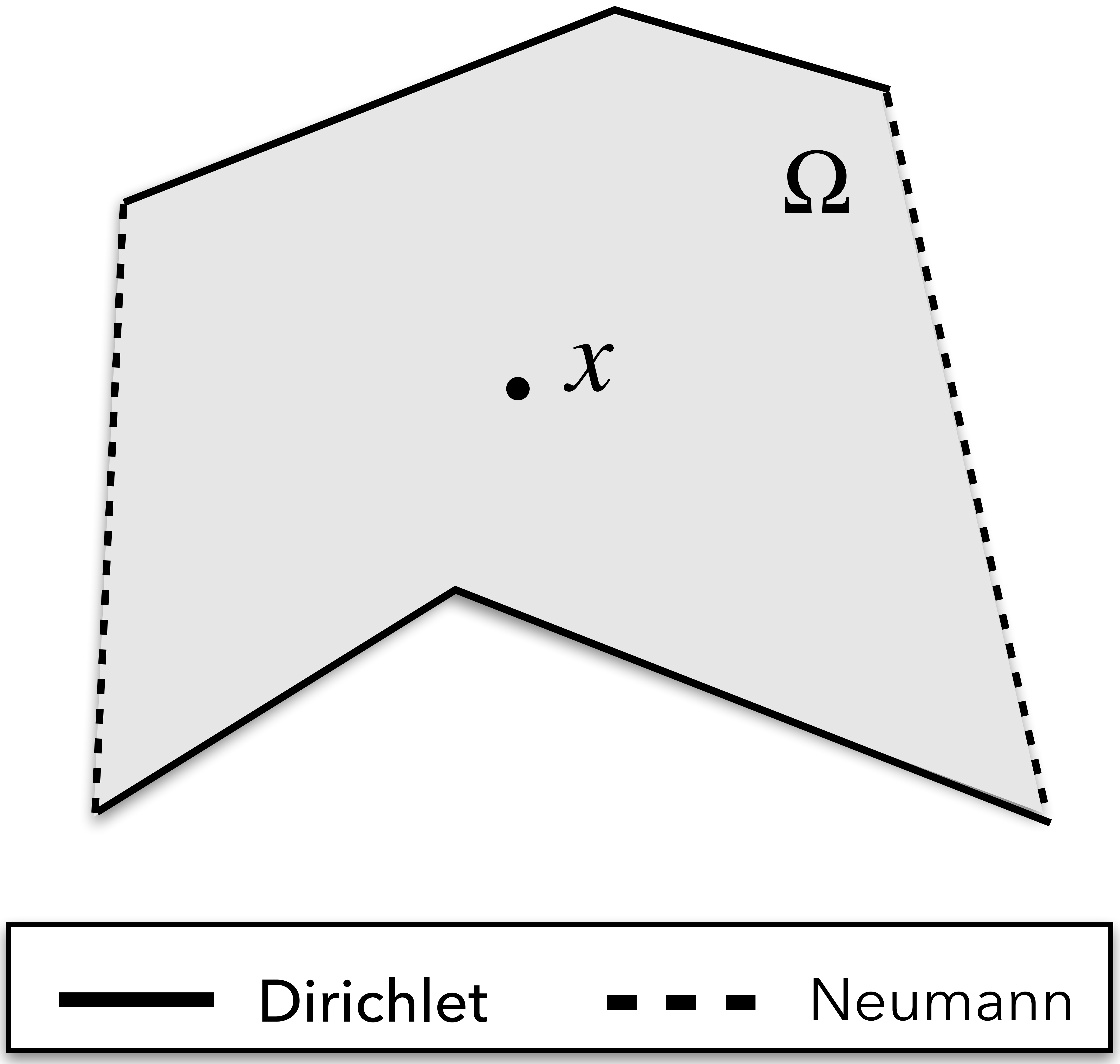

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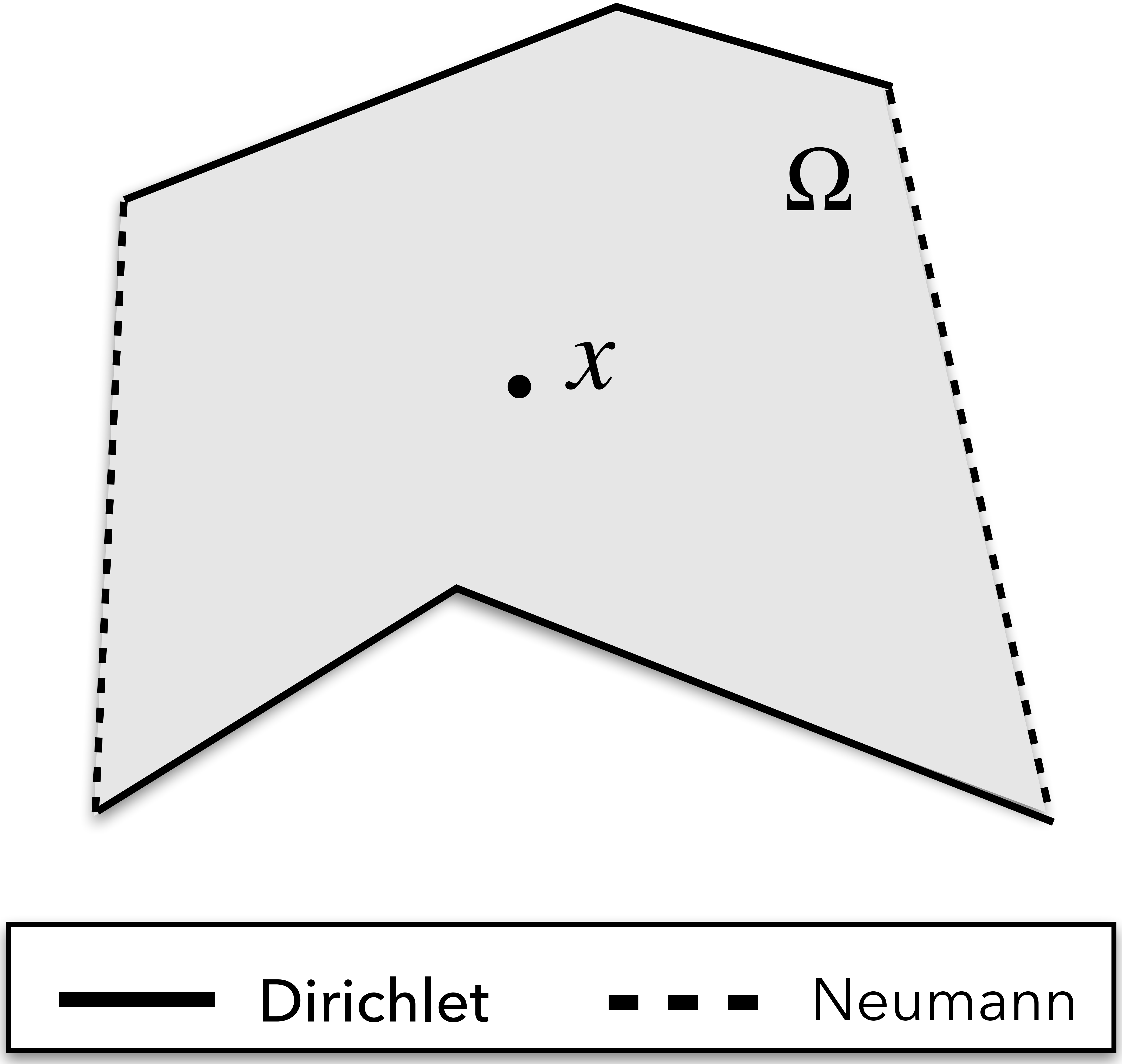

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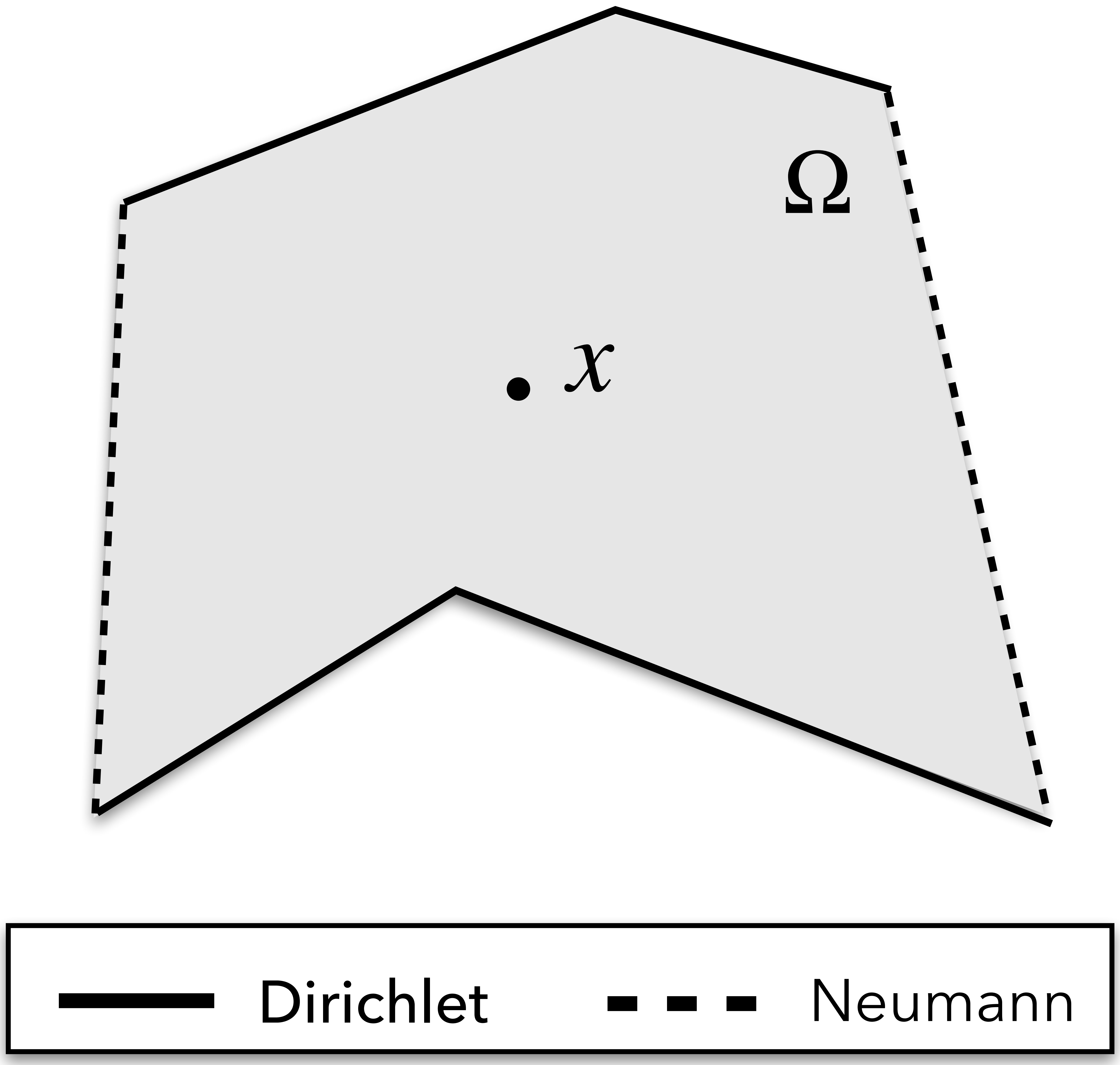

Boundary Integral Equation (BIE)

$$u(x) = \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n} u(y) - G(x,y)\frac{\partial u(y)}{\partial n}\, dy$$

$\Delta u = 0$ on Ω

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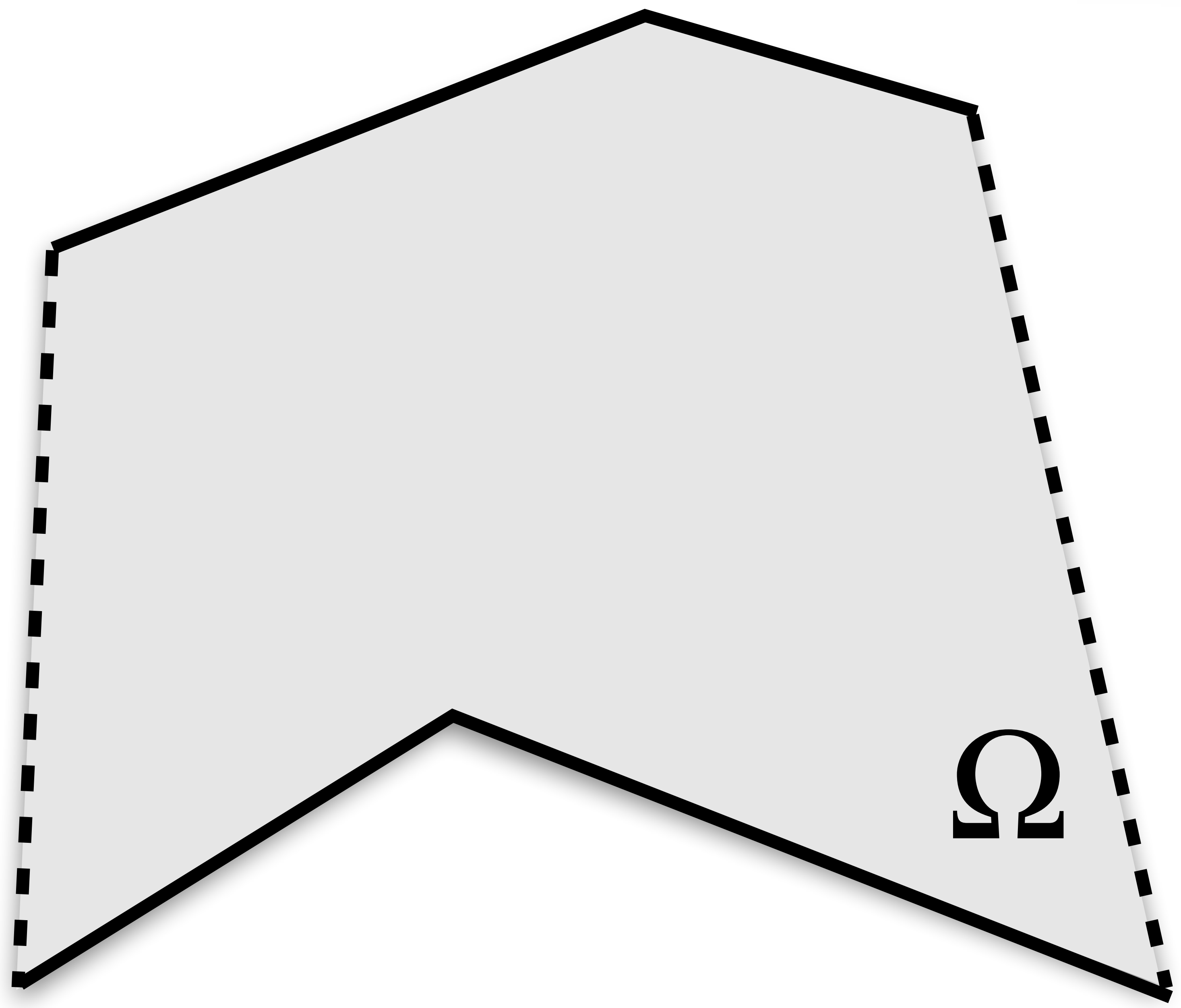

Boundary Integral Equation (BIE)

Dirichlet Data — Neumann Data

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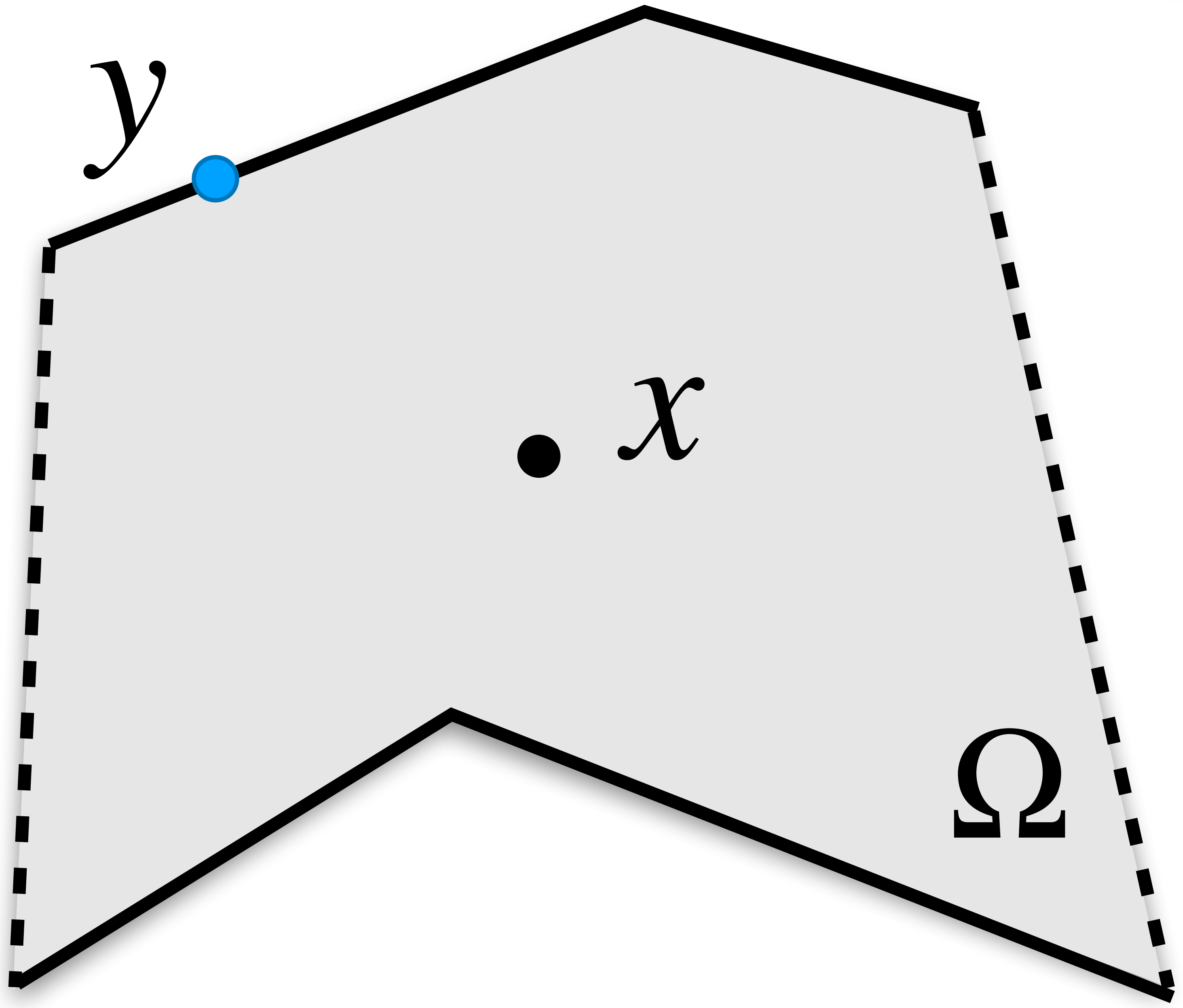

Boundary Integral Equation (BIE)

free-space Poisson kernel — free-space Green kernel

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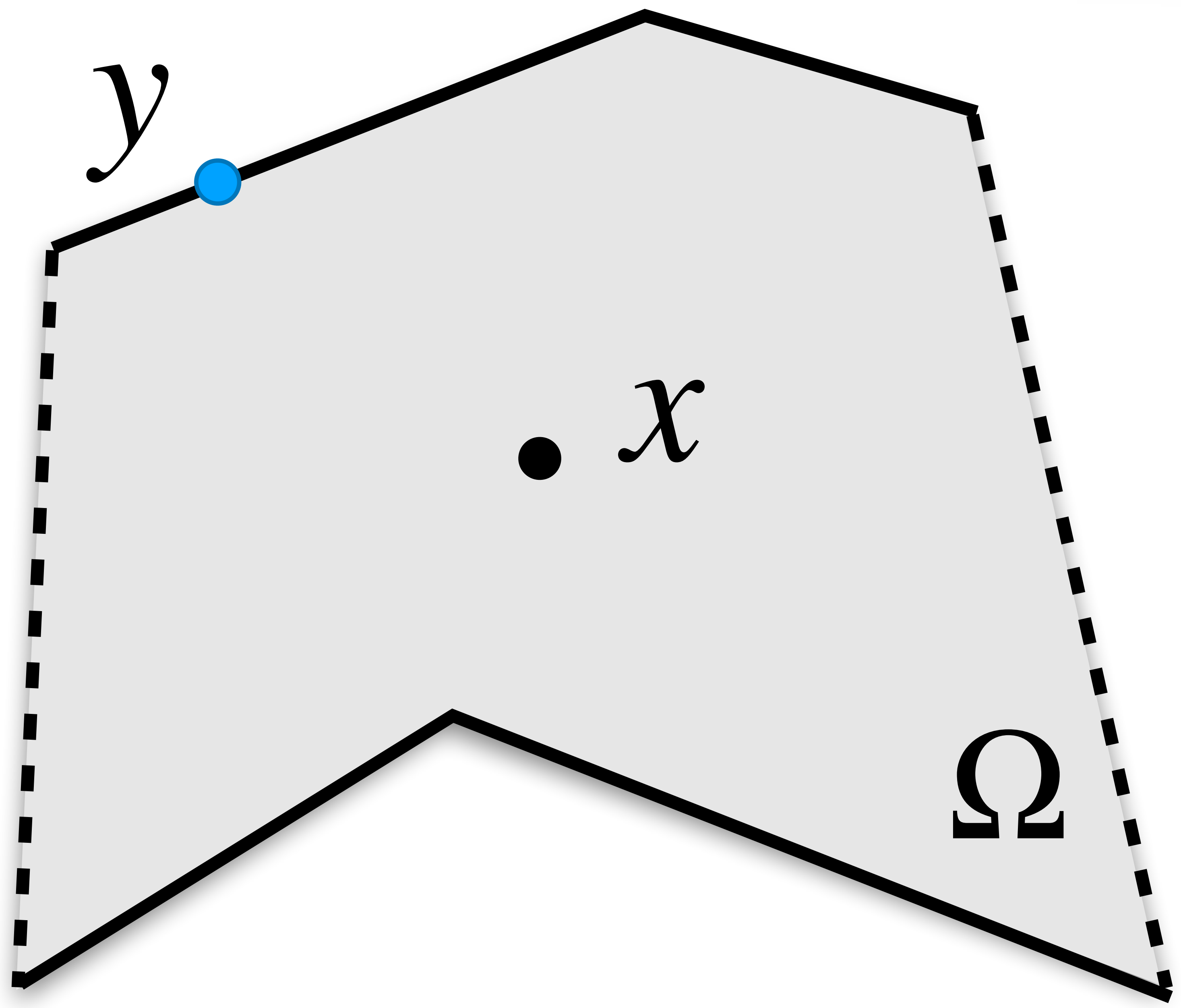

Monte Carlo Estimate of BIE

$$\widehat{u}(x) = \frac{|\partial\Omega|}{N}\sum_{i=1}^{N}\frac{\partial G(x,y_i)}{\partial n}u(y_i) - G(x,y_i)\frac{\partial u(y_i)}{\partial n}$$

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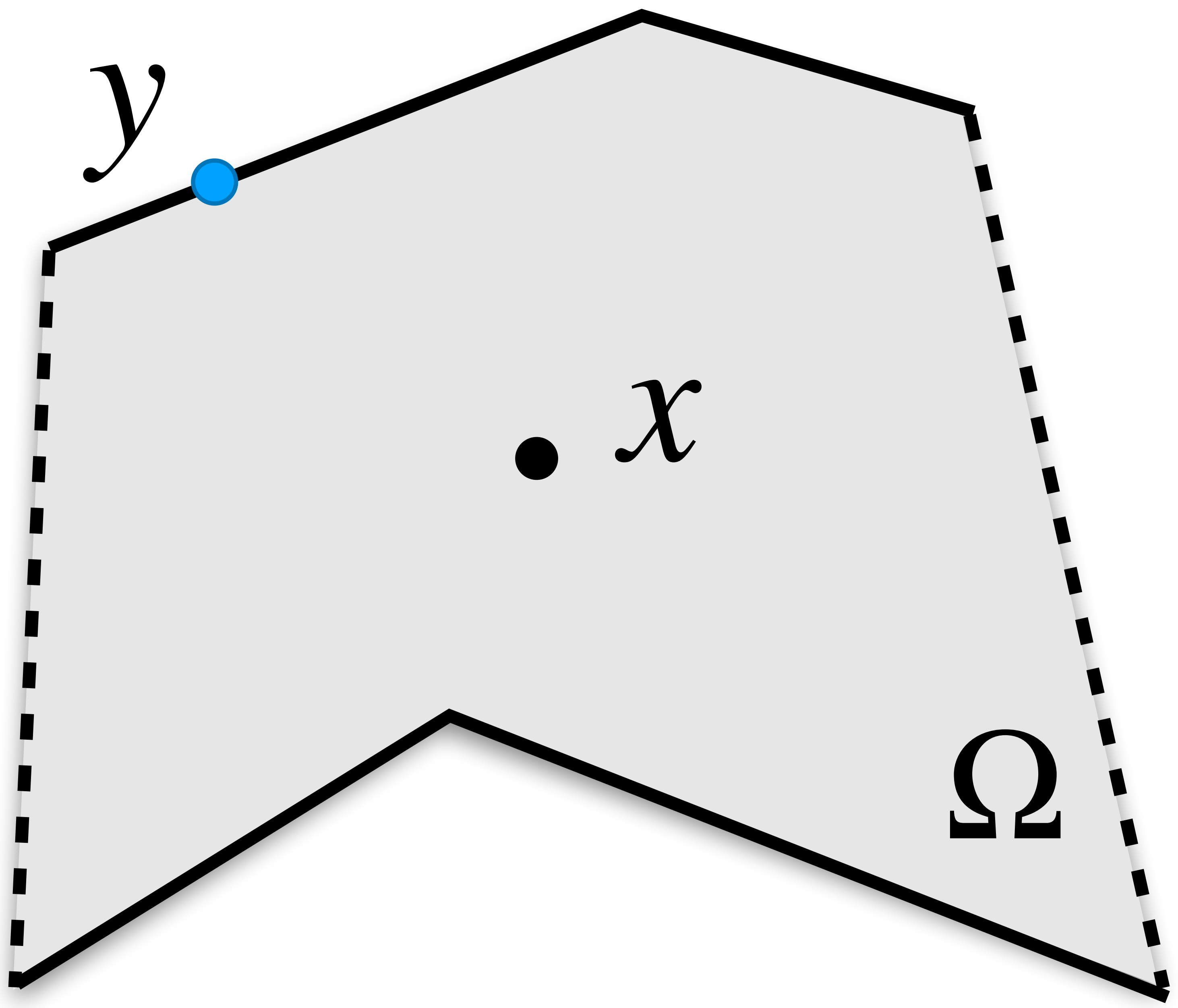

Monte Carlo Estimate of BIE

known　　requires estimate

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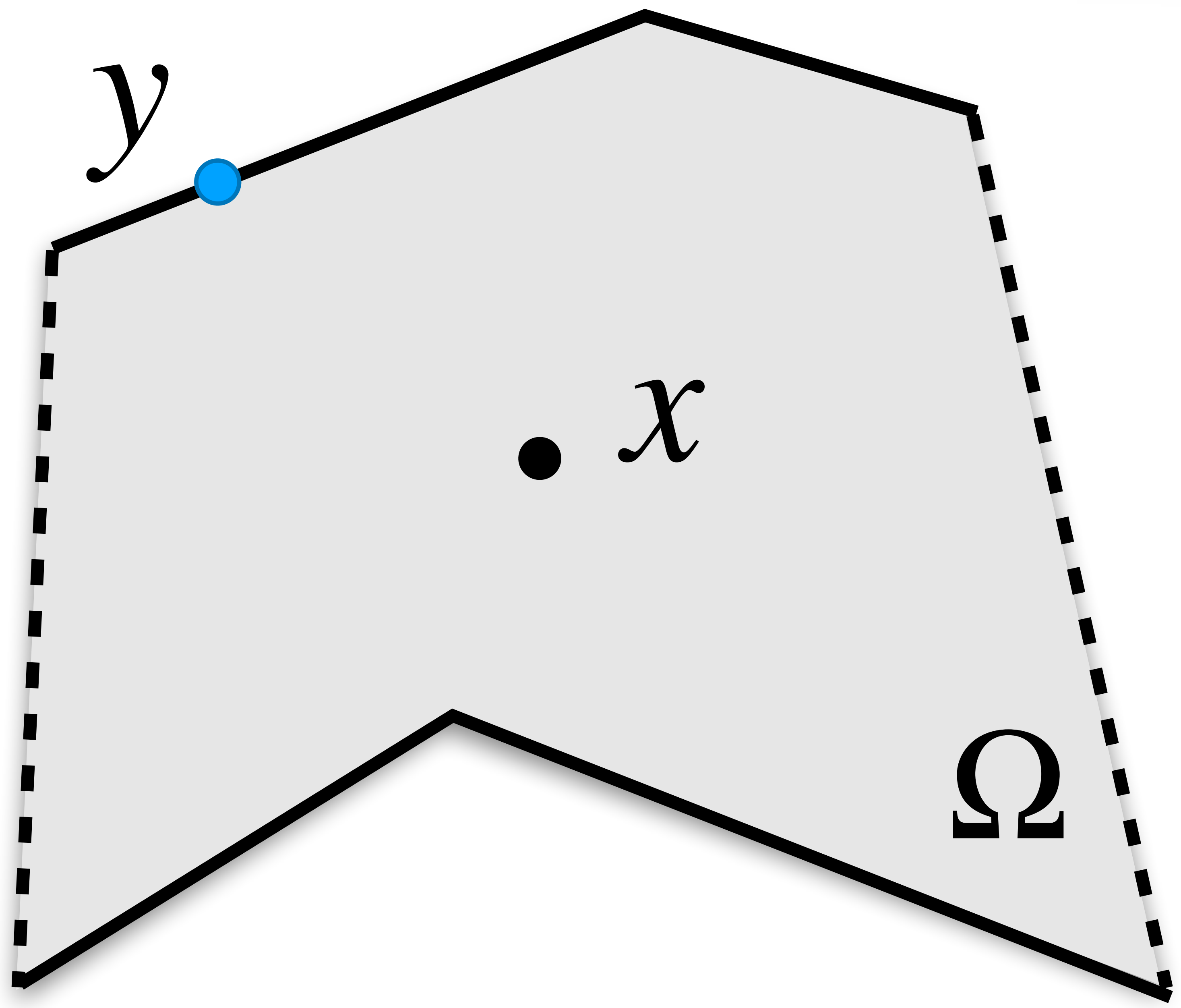

Monte Carlo Estimate of BIE

requires estimate — known

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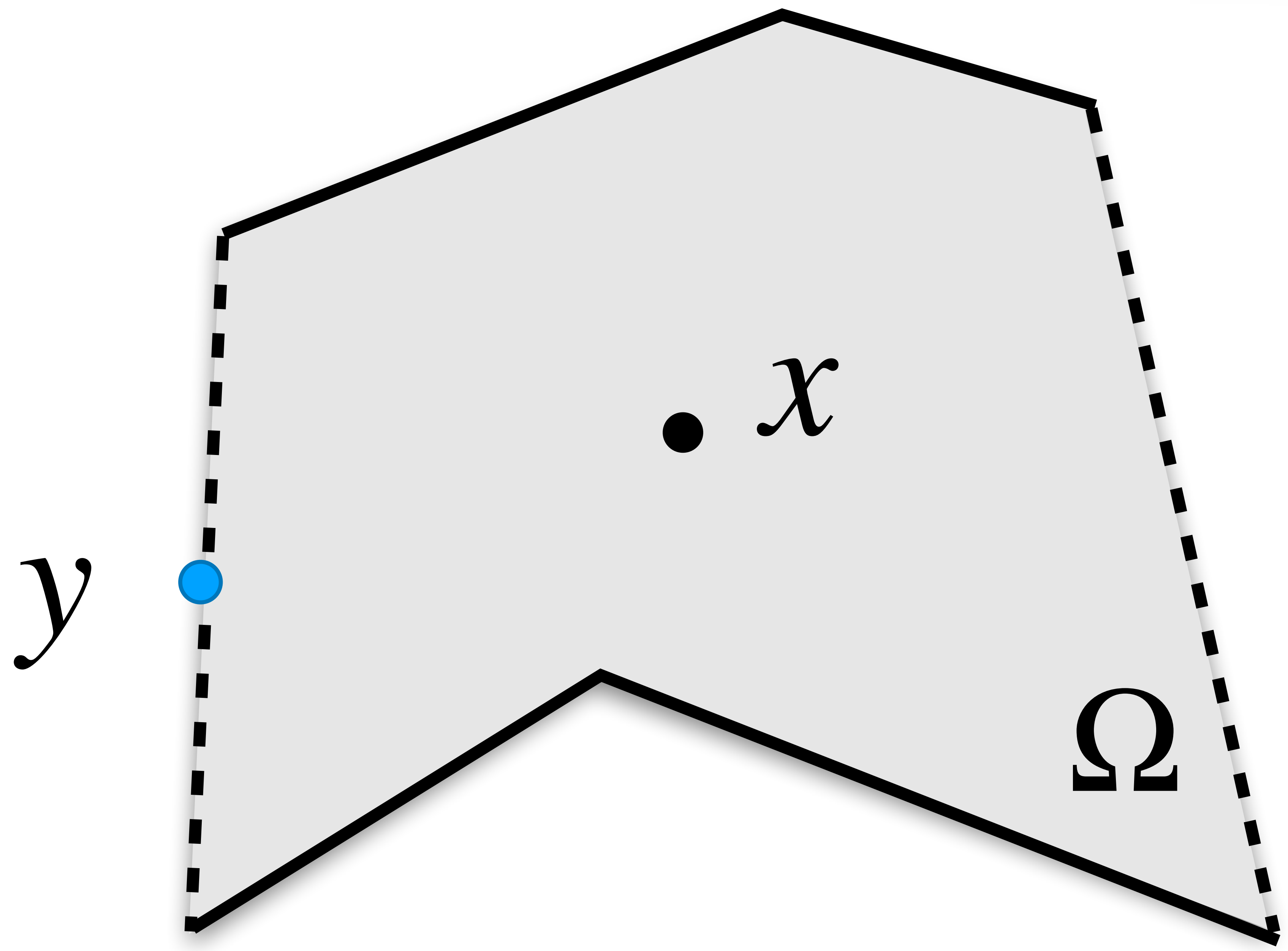

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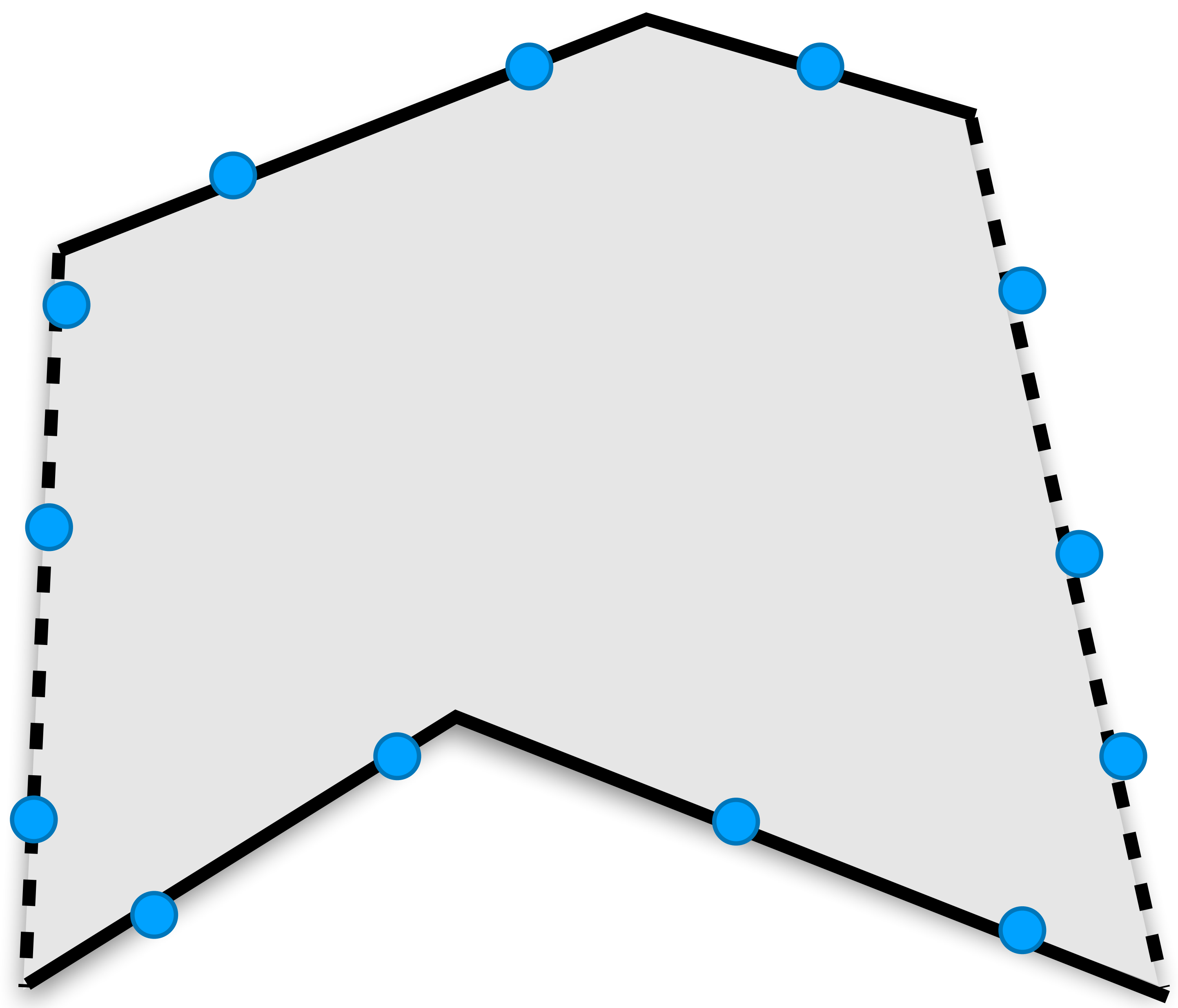

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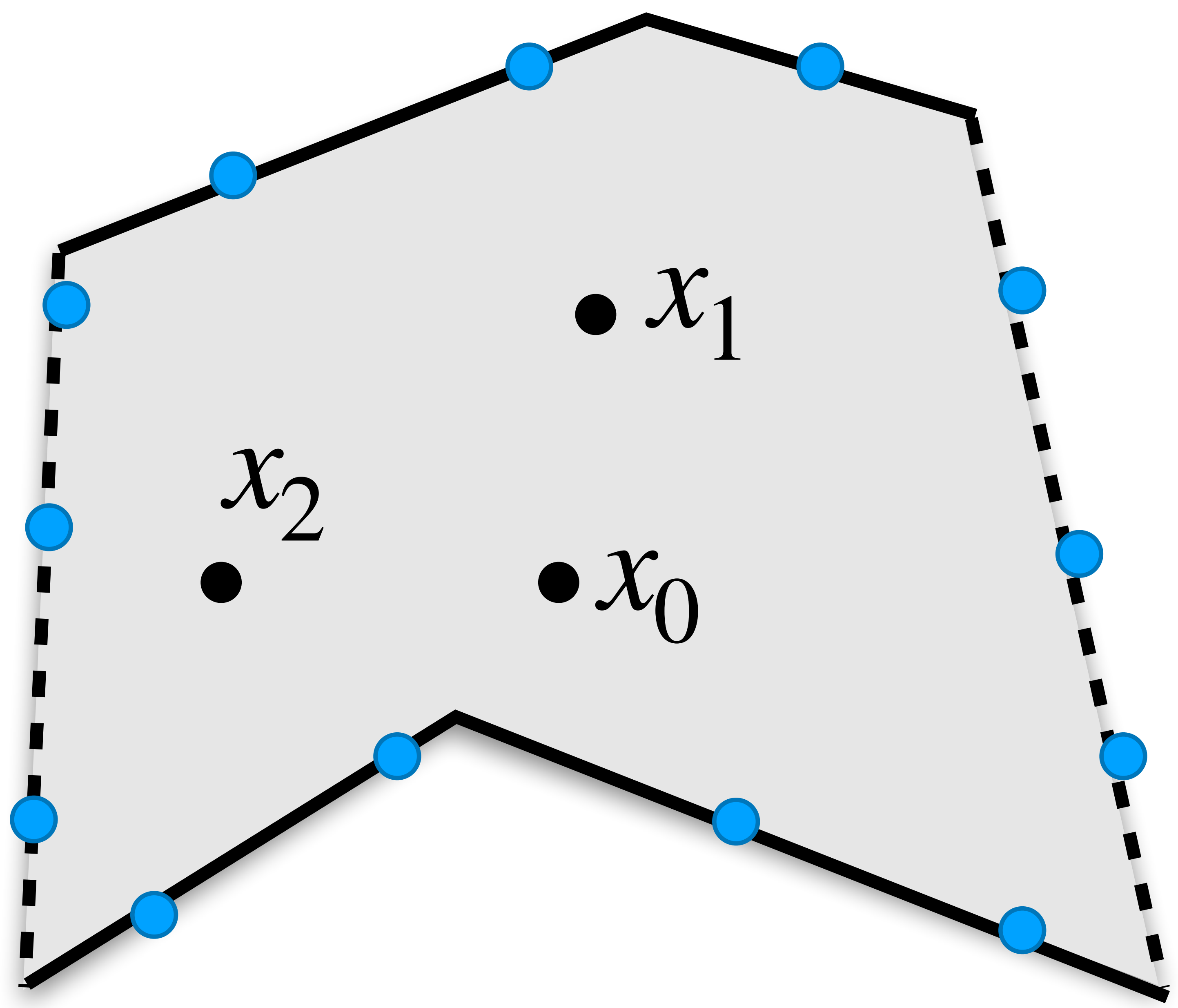
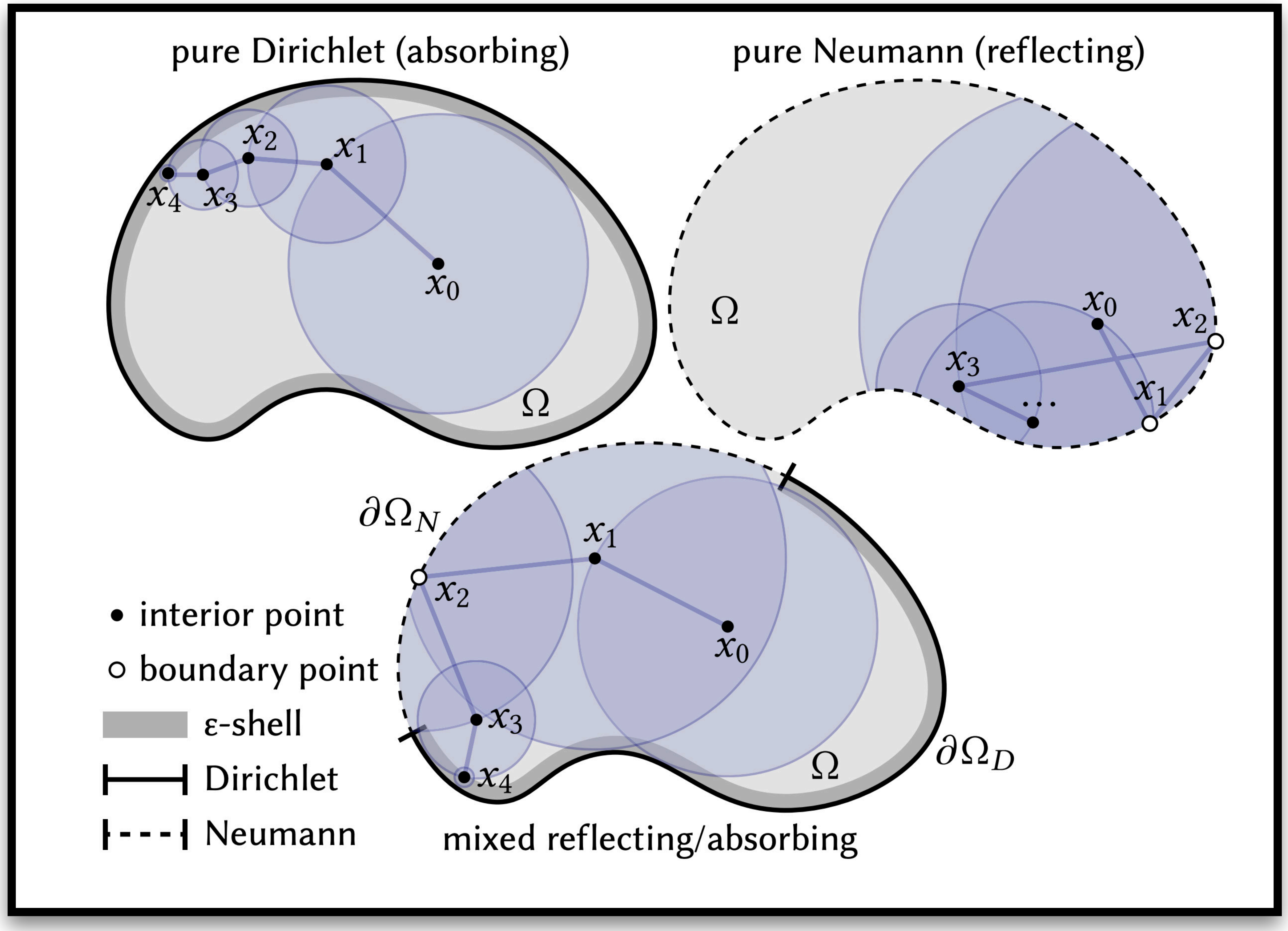


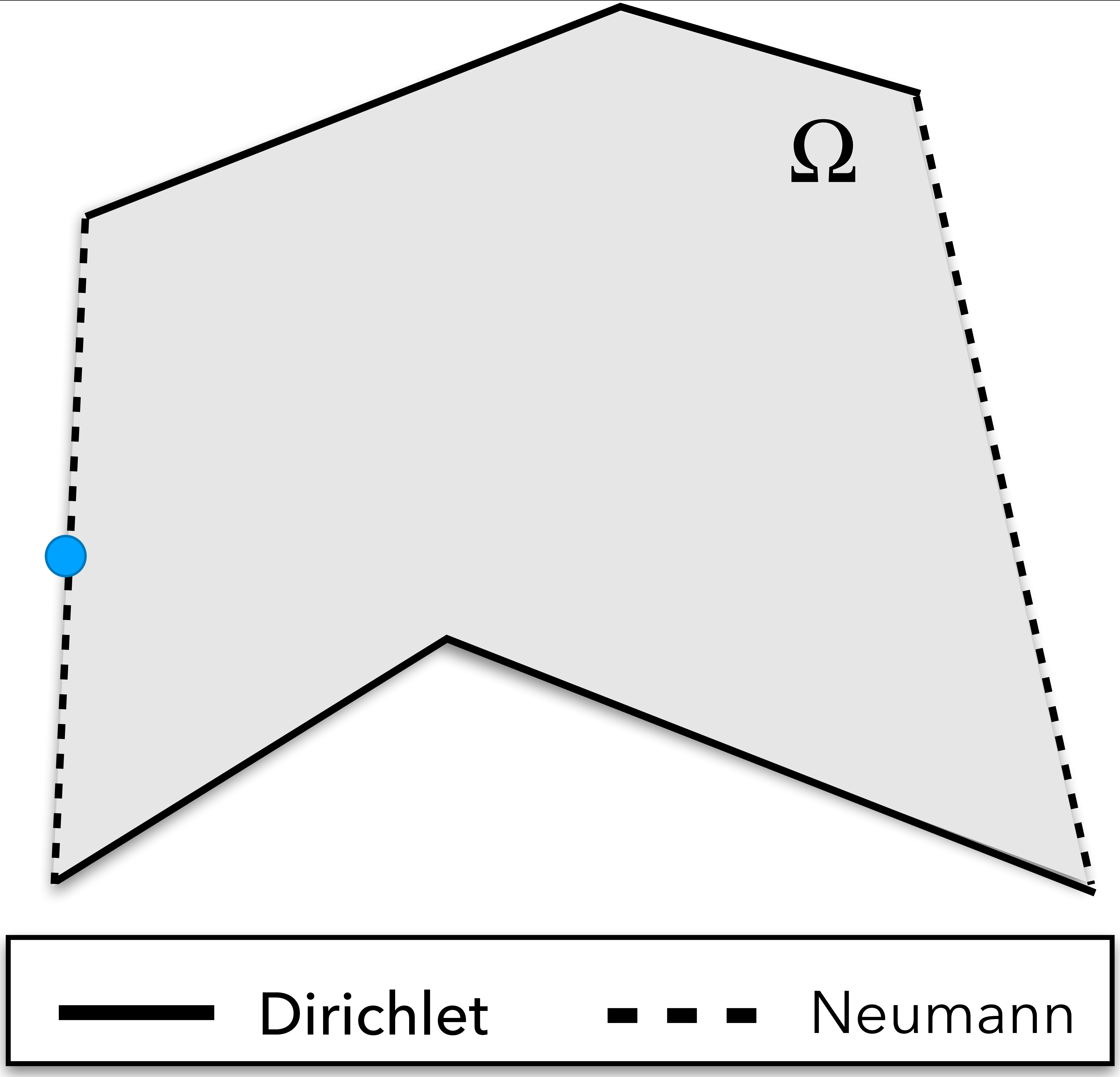
we can reuse boundary value estimates!

METHOD

Estimating Dirichlet Boundary Values $\hat{u}$

Walk on Stars [Sawhney et al. 2023]:





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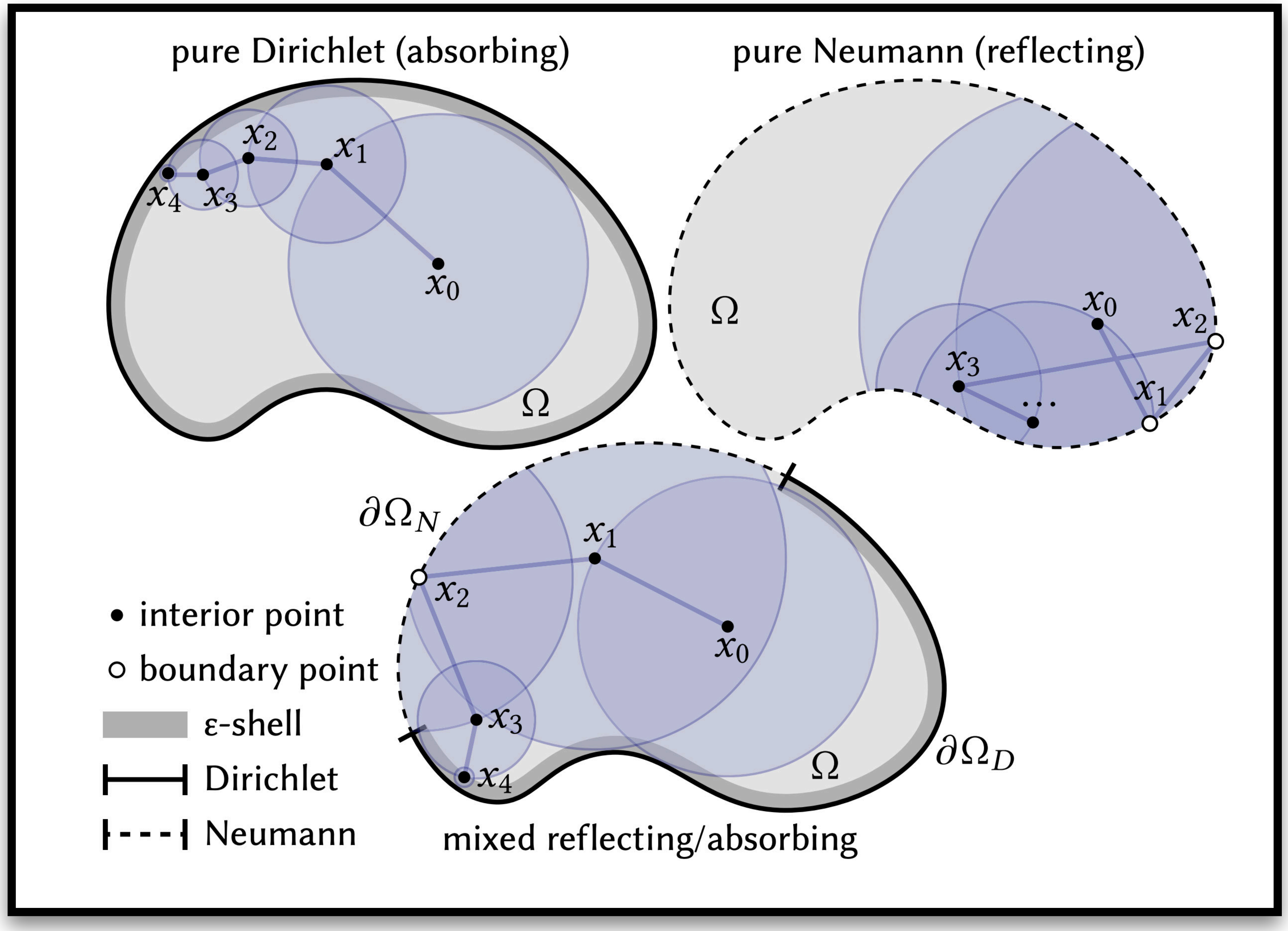
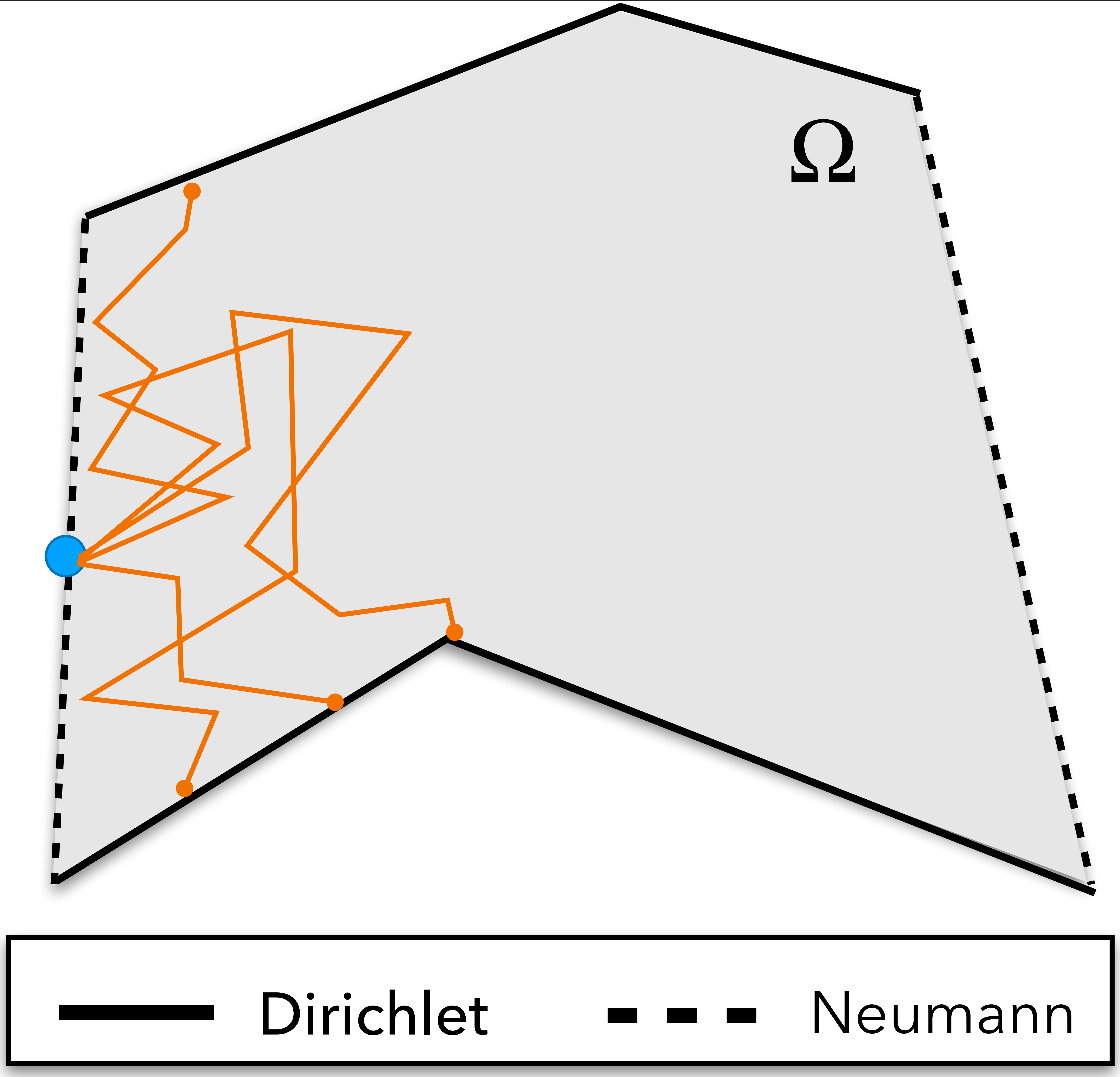




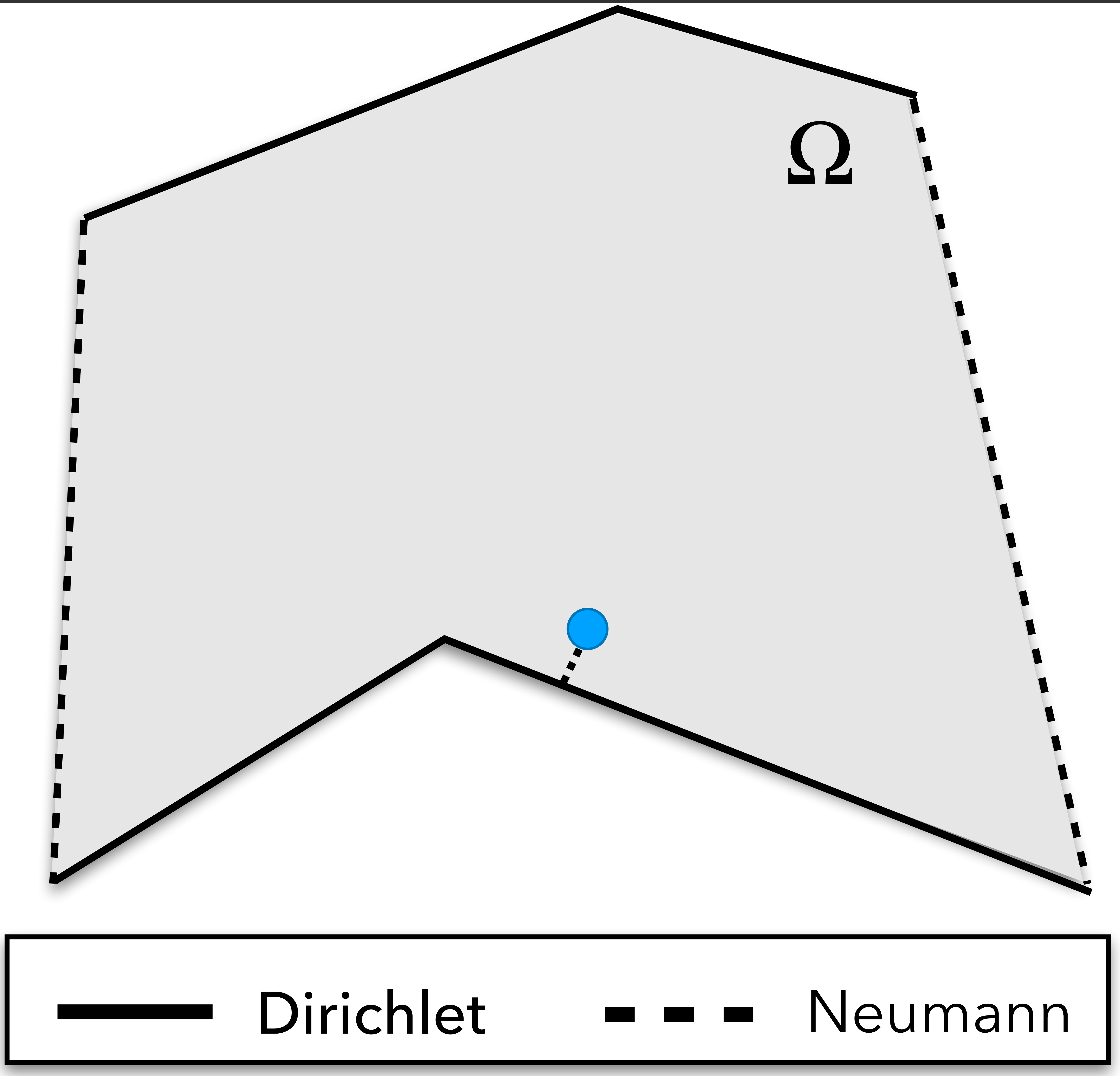
Estimating Neumann Boundary Values $d\hat{u}/dn$

Spatial derivative **inside a ball** [Sawhney & Crane 2020]:

$$\nabla_x u(x) = \frac{1}{|B|}\int_{\partial B} u(y)\ v(y)\ dy$$

Normal derivative **on the boundary**:

$$\frac{du(x)}{dn_x} = n_x \cdot \nabla_x u(x)$$



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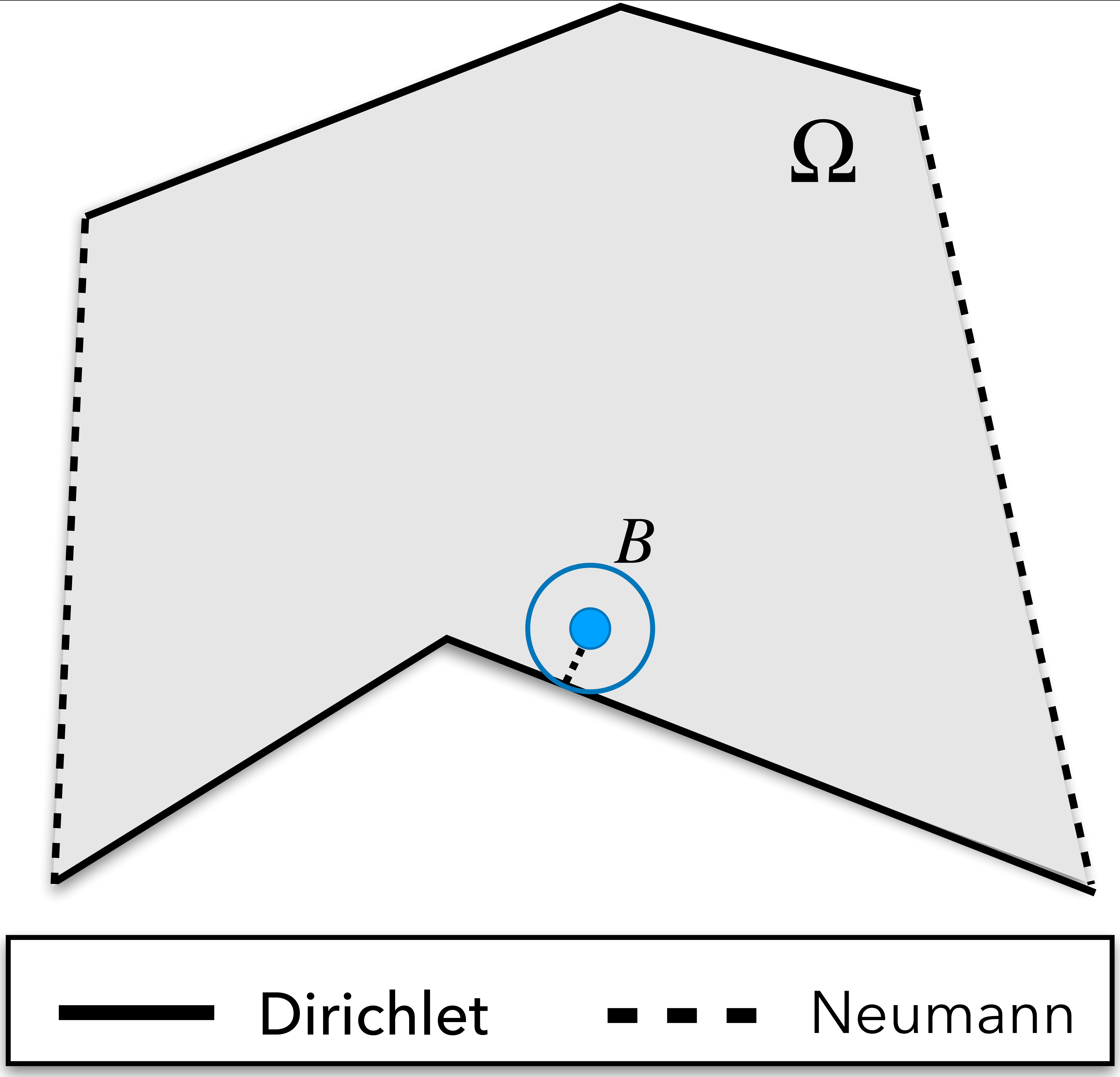

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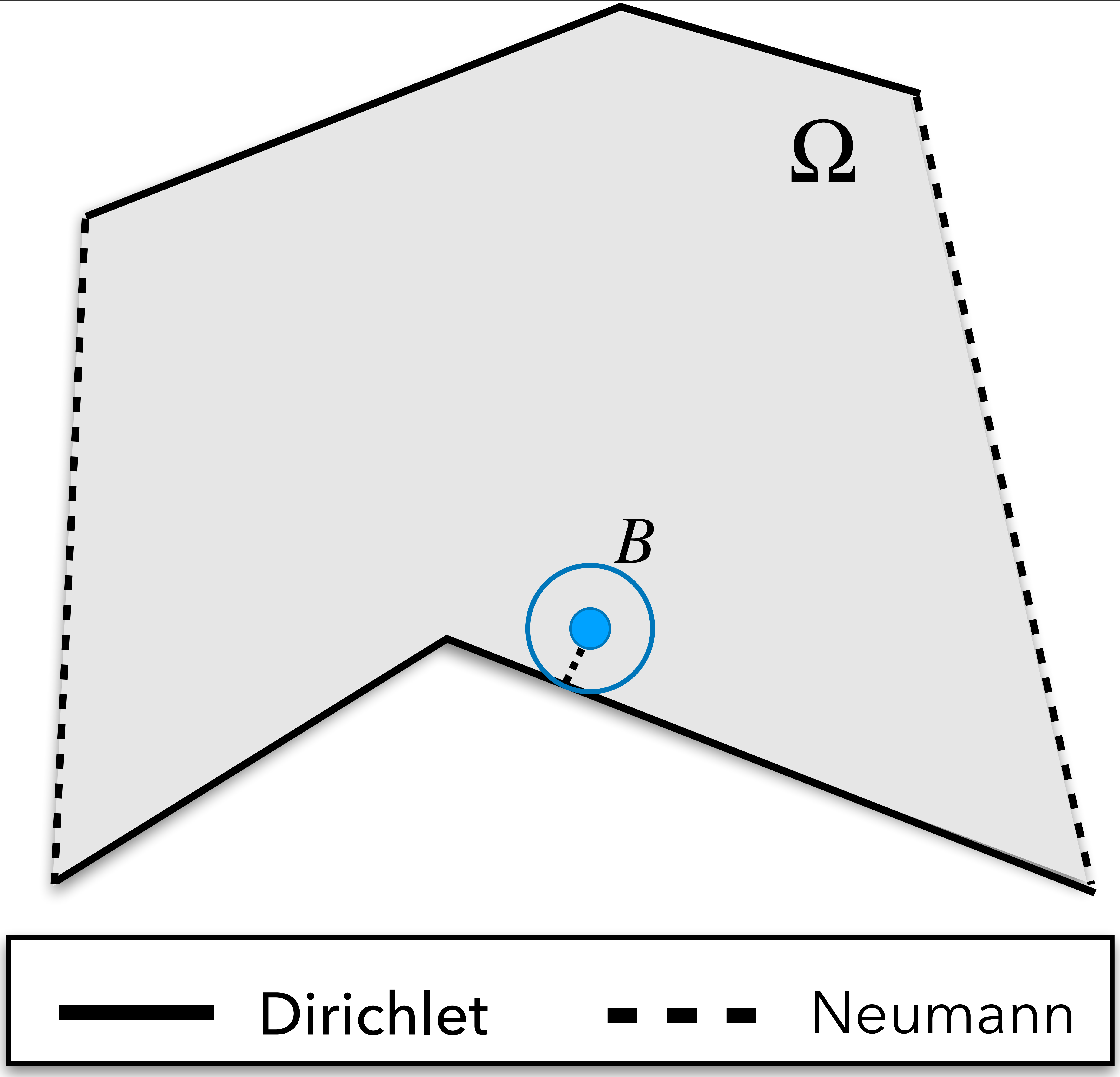

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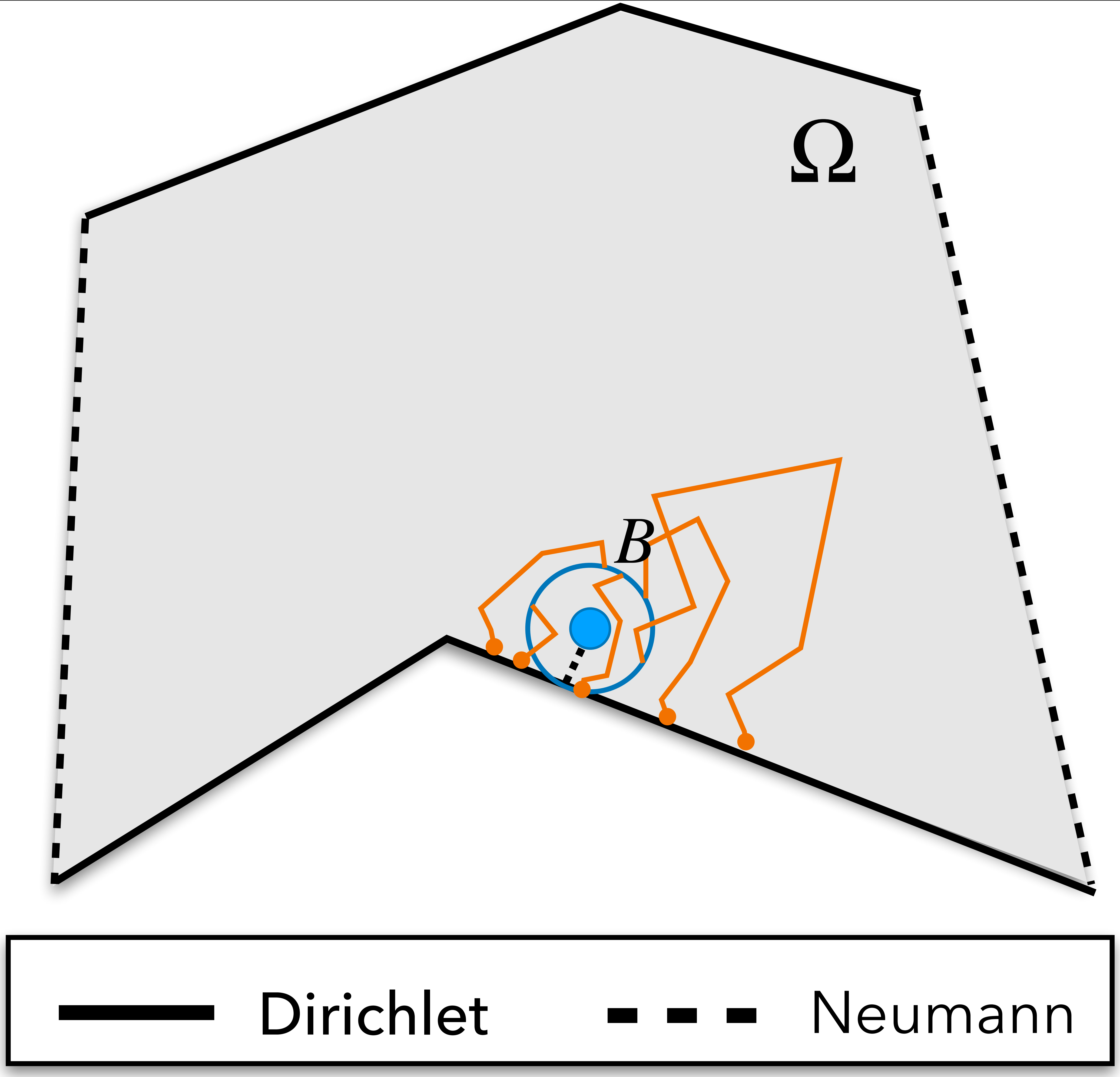
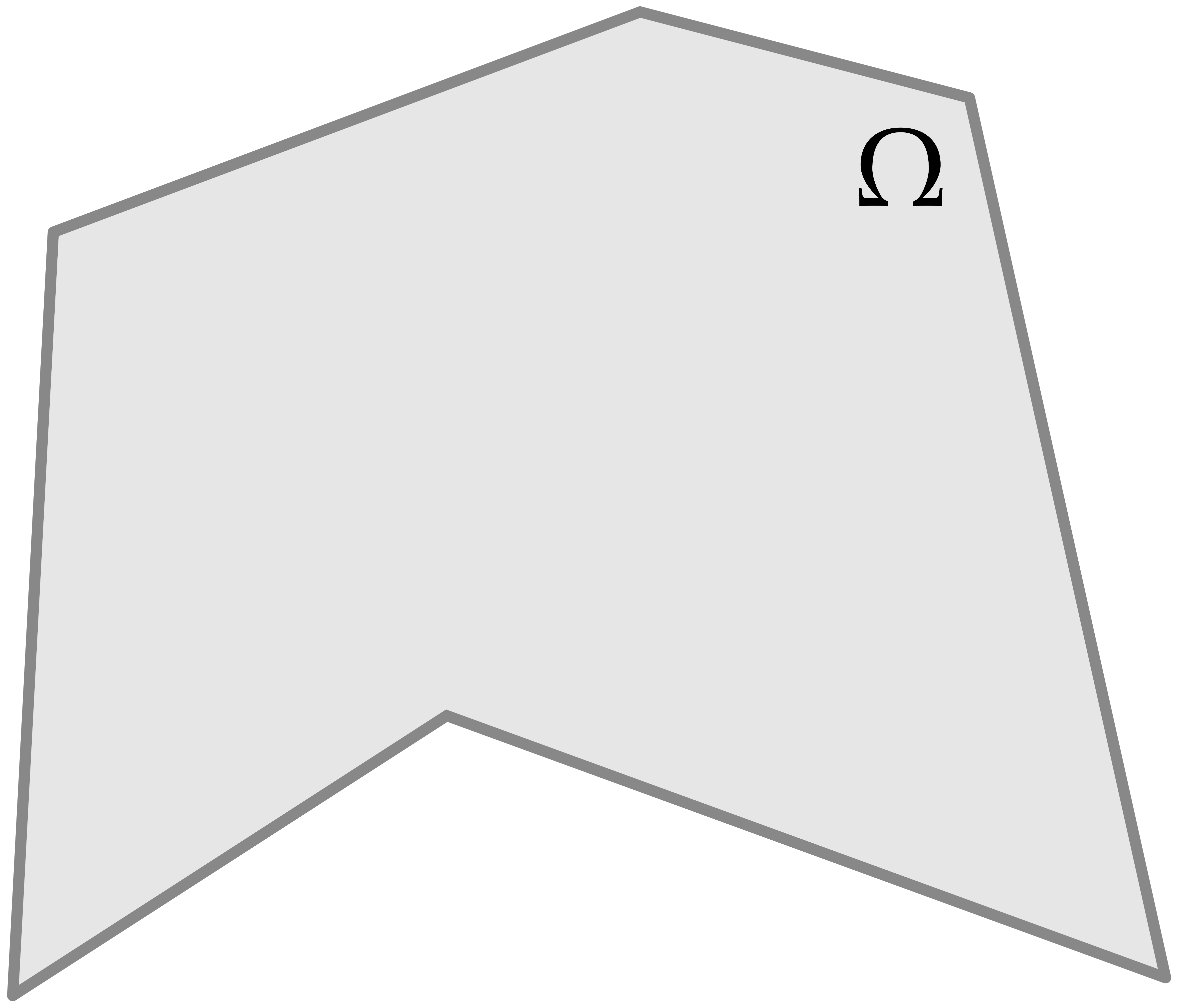


Boundary Value Caching (BVC)



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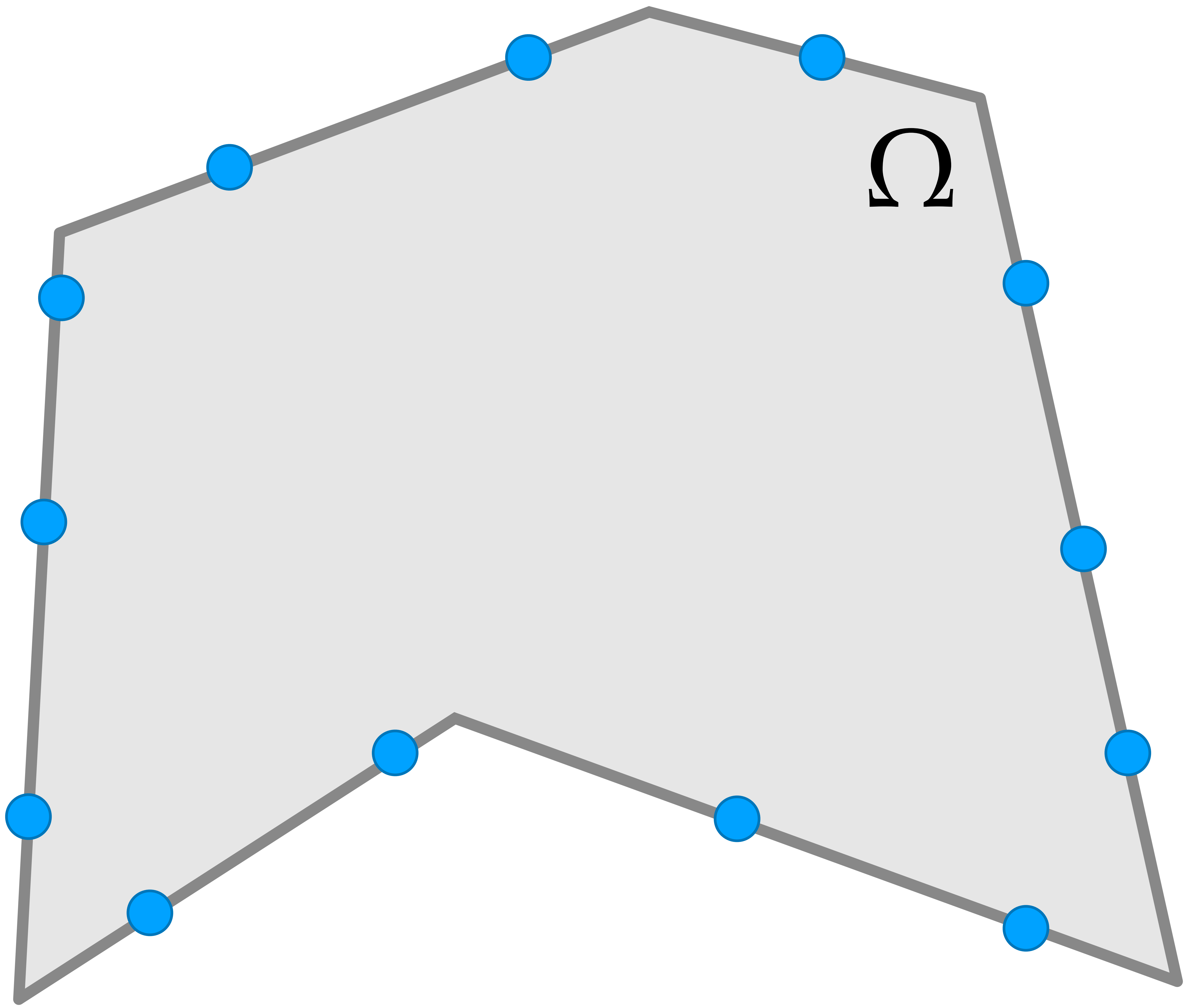

generate samples on boundary $\partial\Omega$

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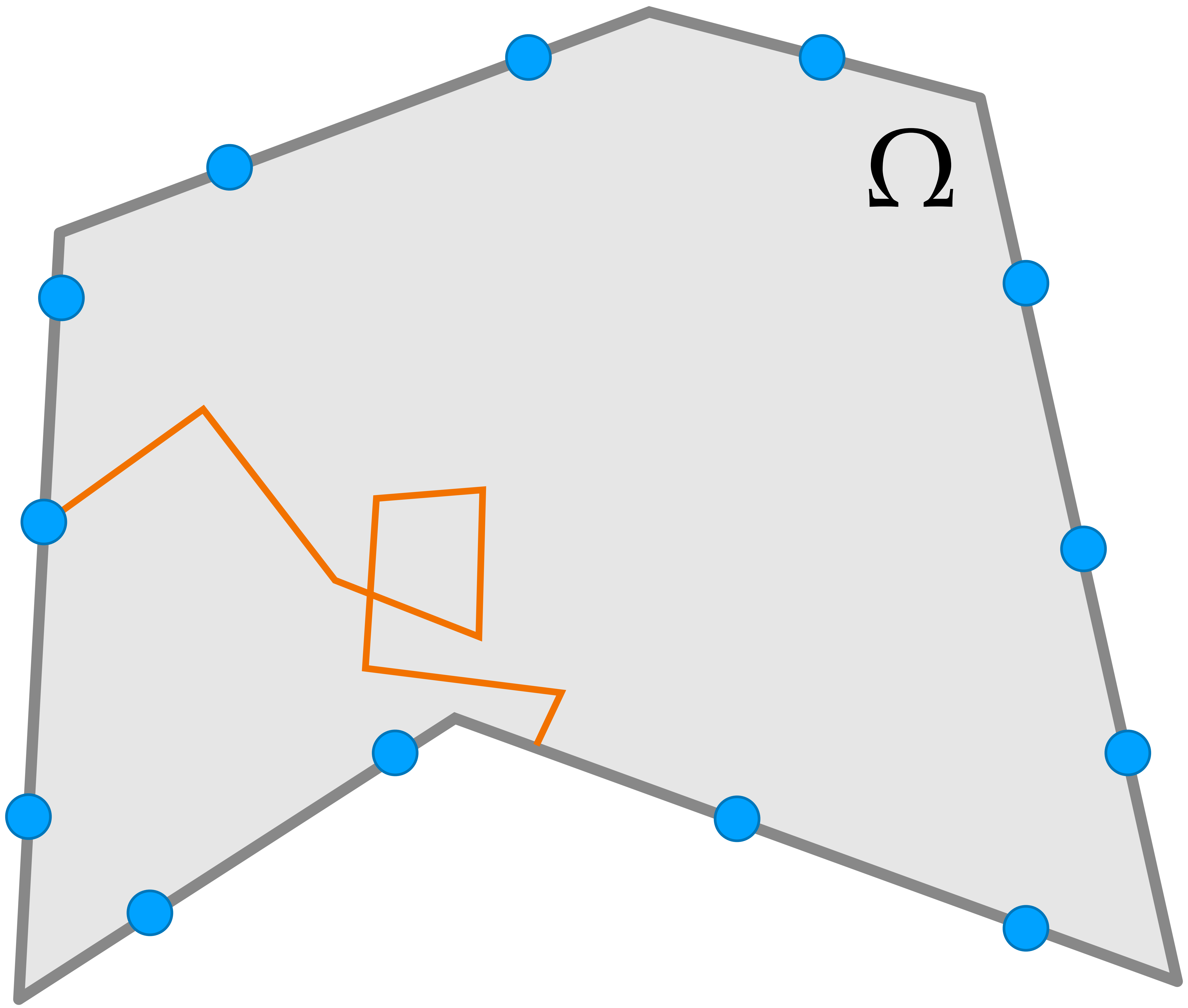

generate samples on boundary $\partial\Omega$

↓

use WoSt to estimate u & $\frac{du}{dn}$

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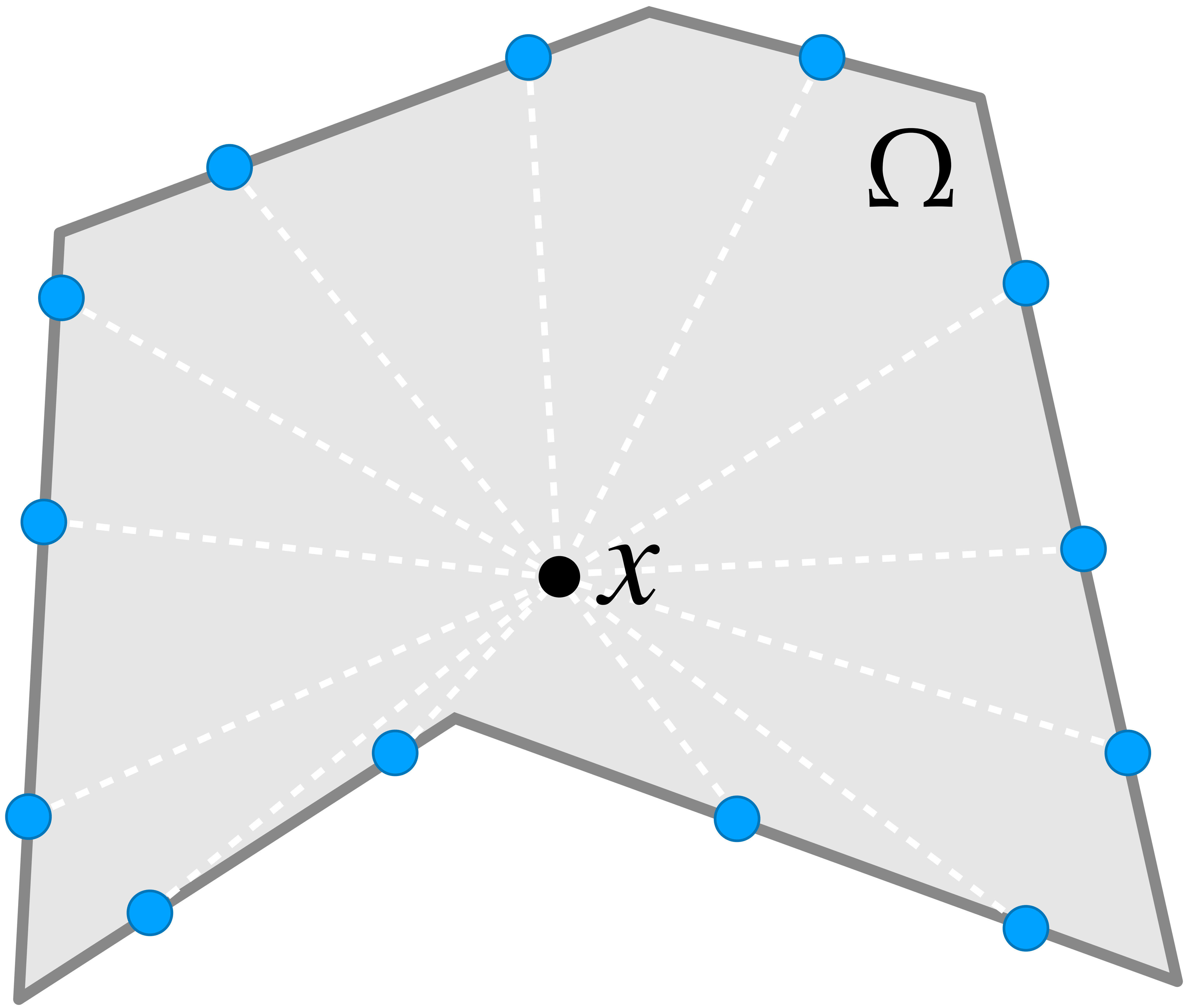


generate samples on boundary $\partial\Omega$

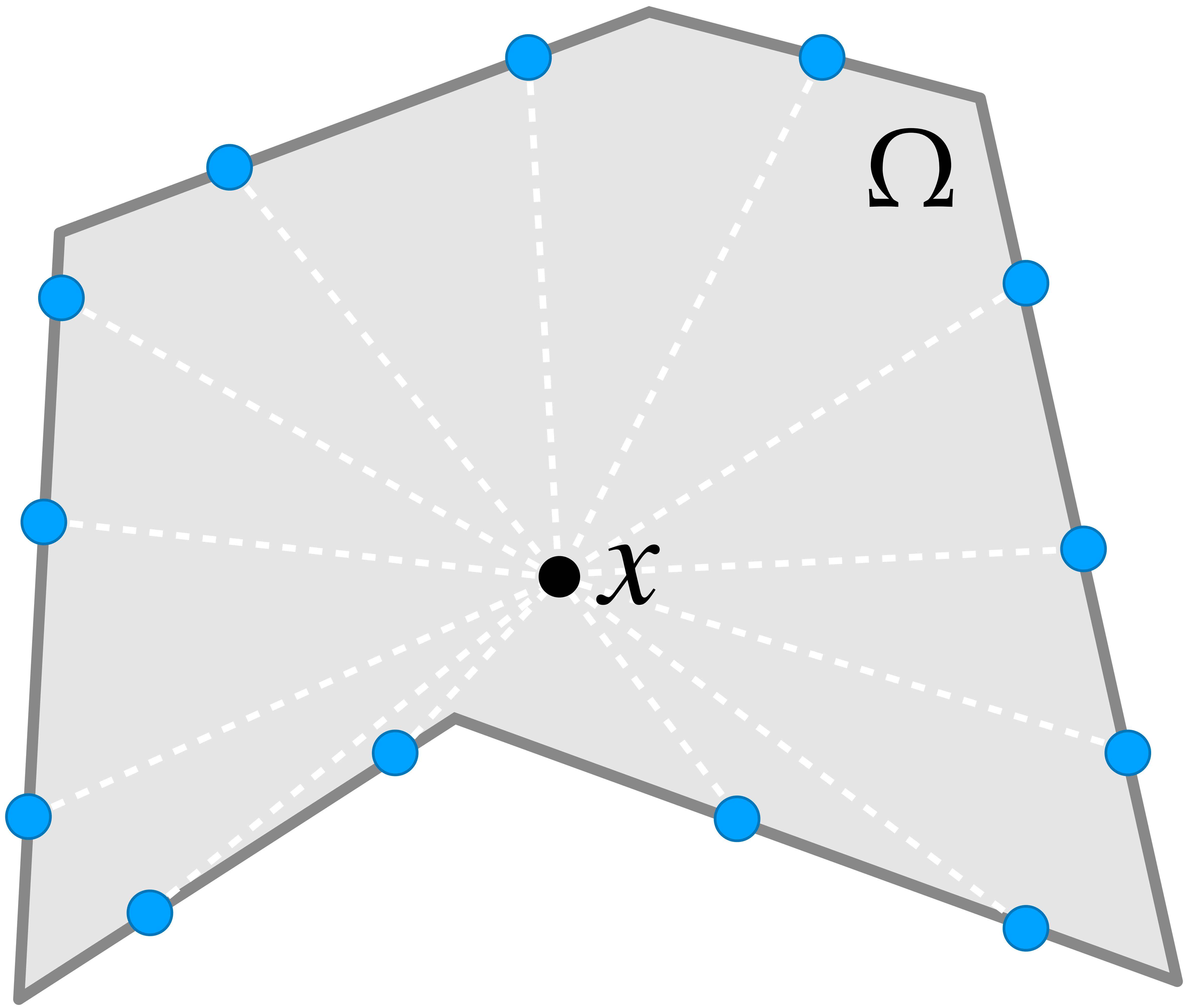
↓

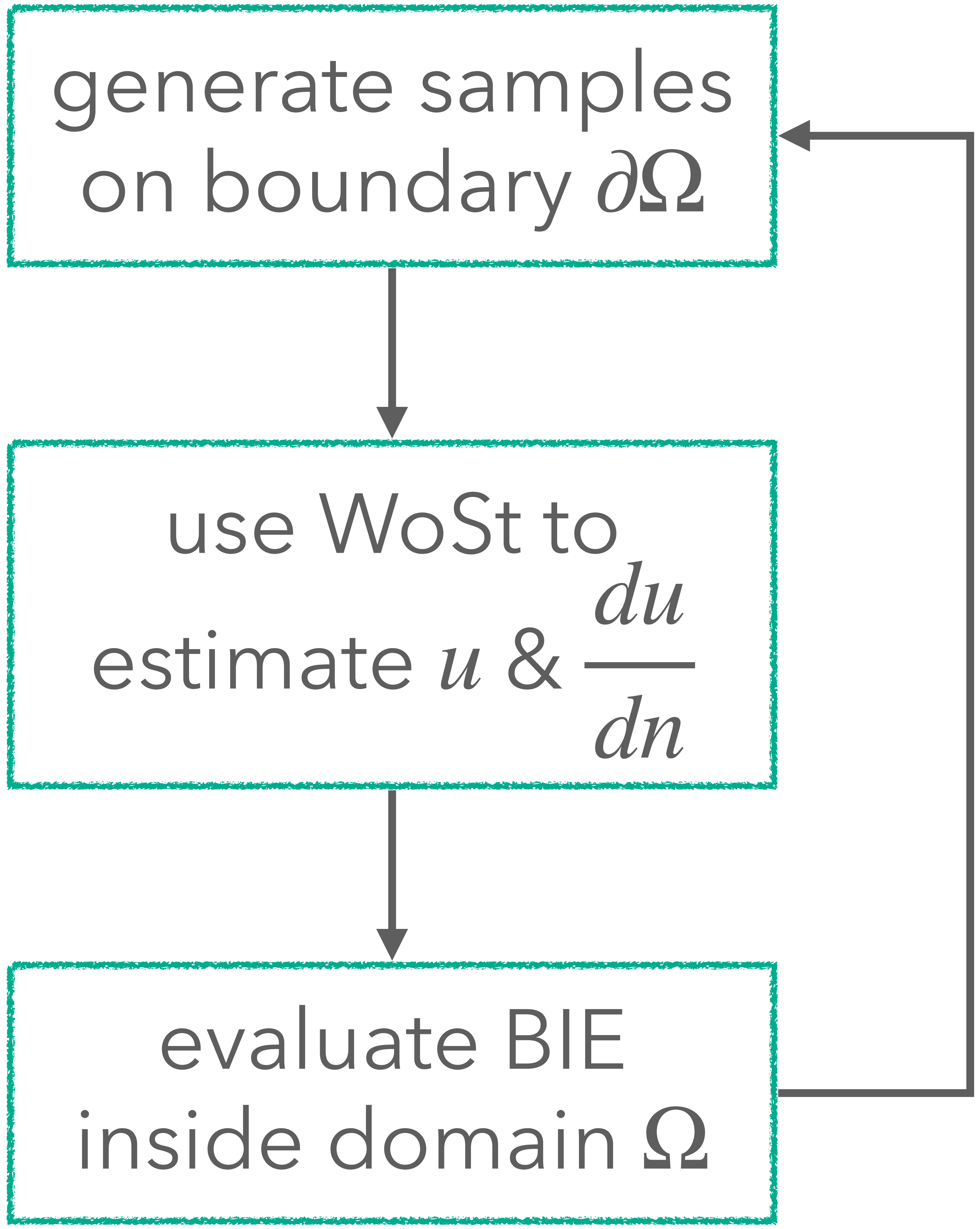
use WoSt to estimate u & $\frac{du}{dn}$

↓

evaluate BIE inside domain Ω

Boundary Value Caching (BVC)





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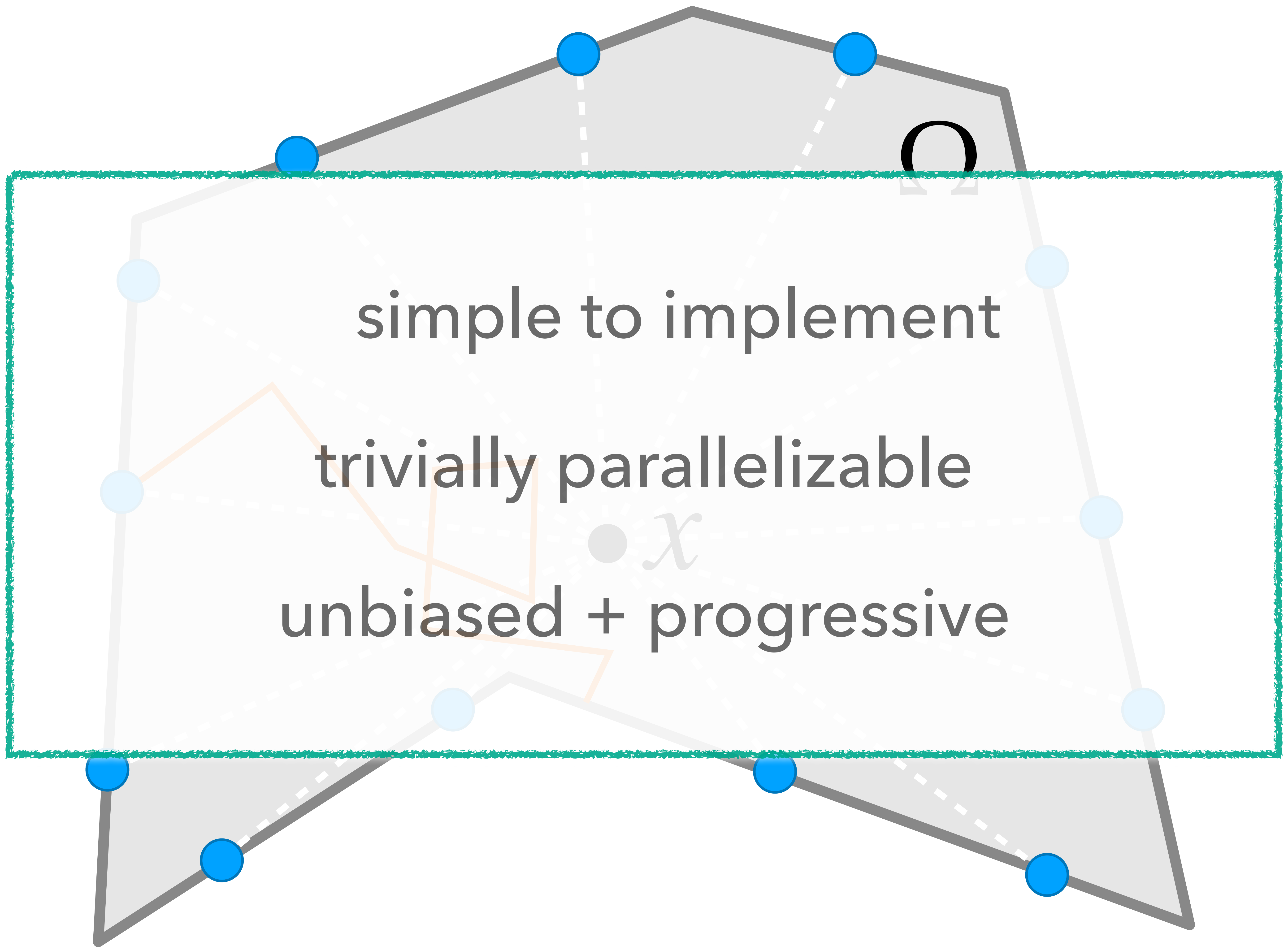
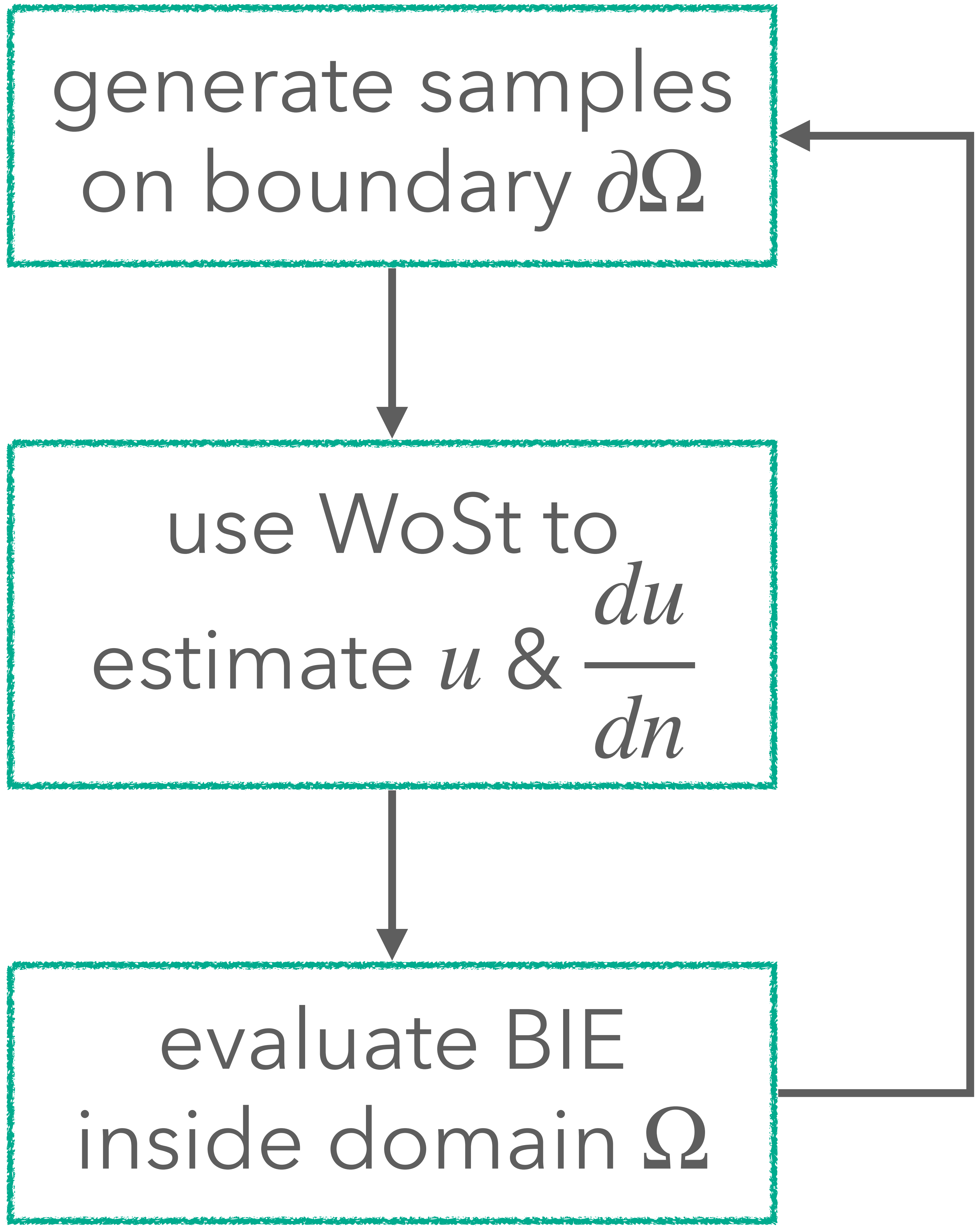


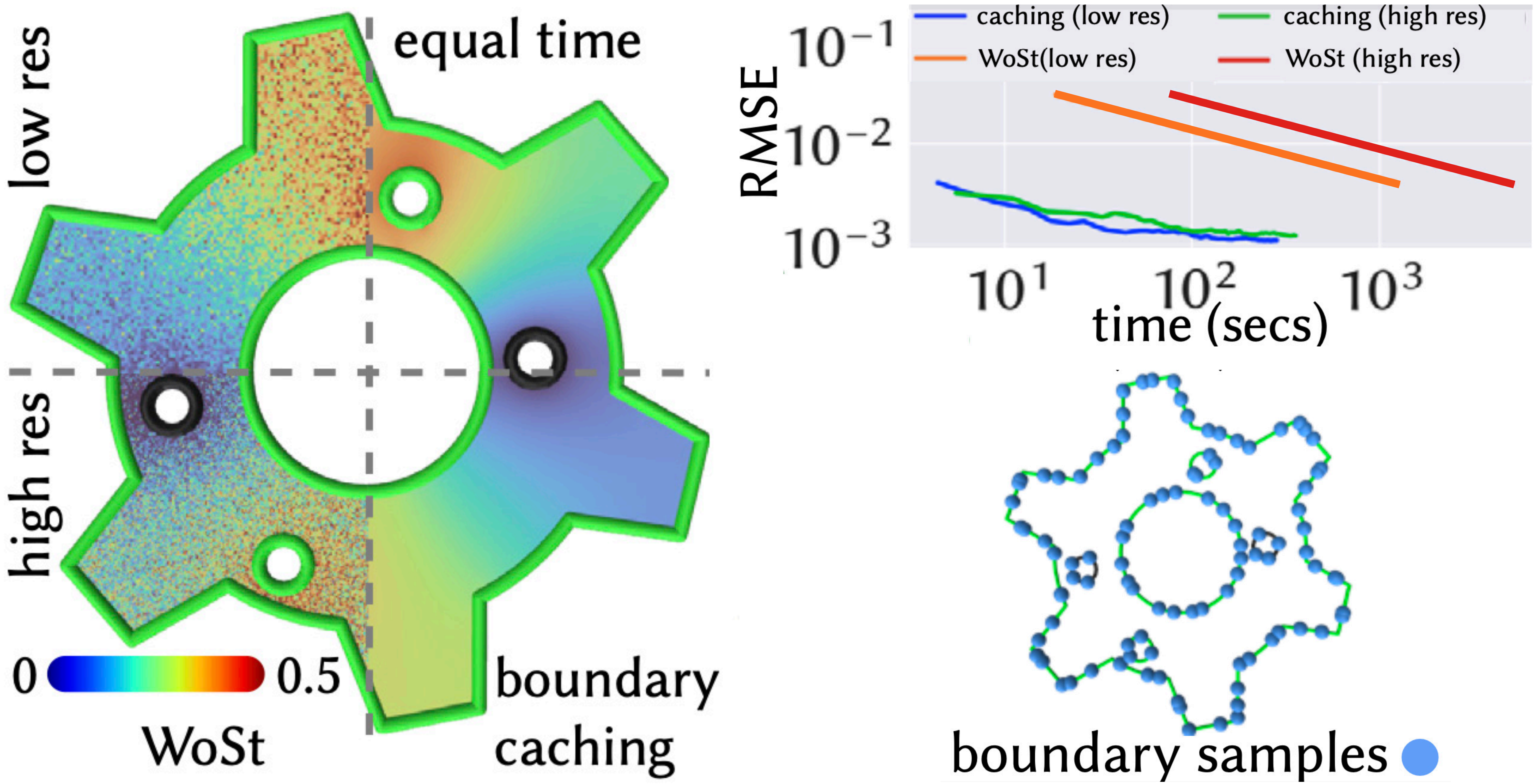
simple to implement

trivially parallelizable

unbiased + progressive

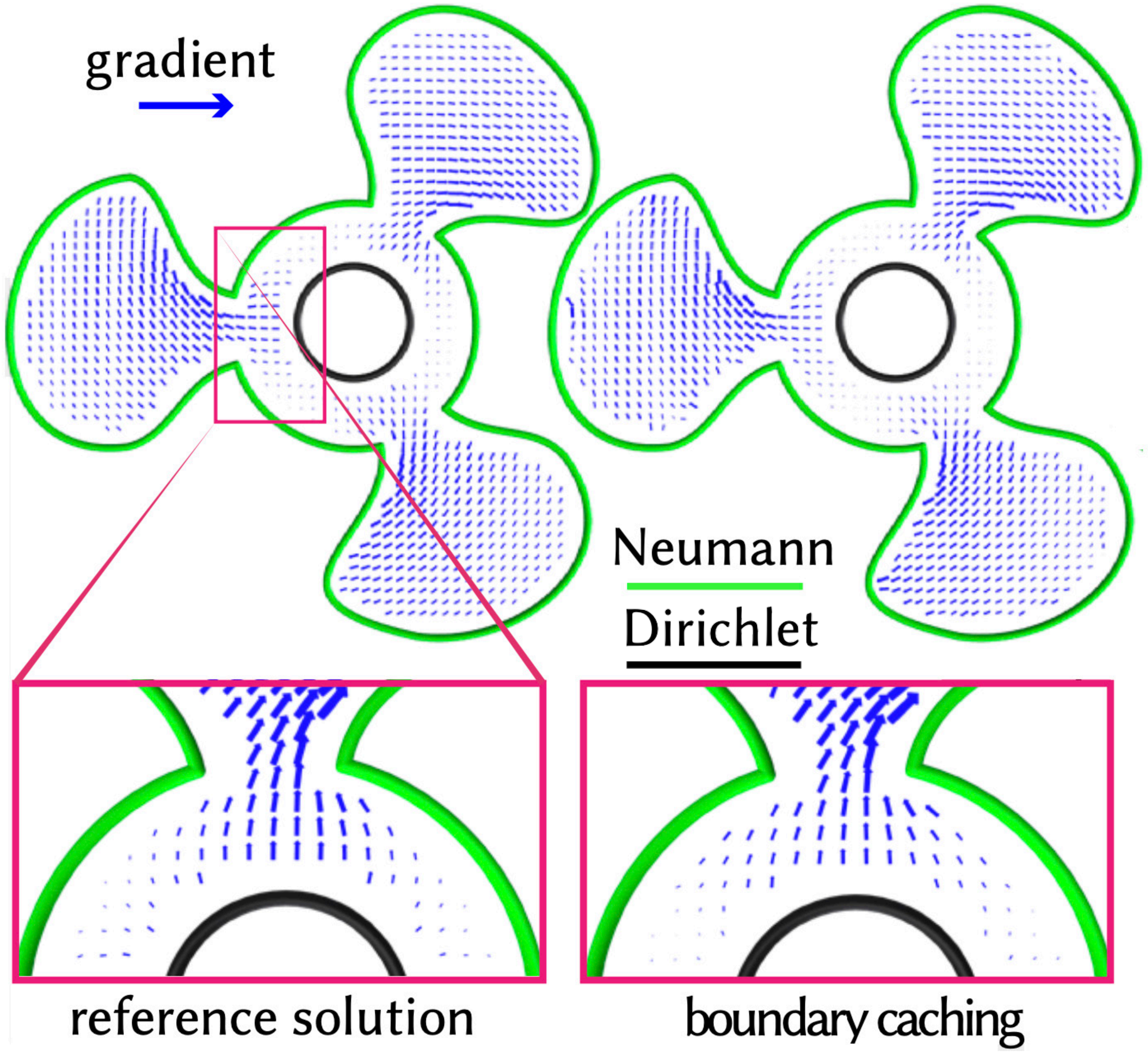


Solution Estimates with BVC



Gradient Estimates with BVC

Can reuse **same** boundary cache for gradients!



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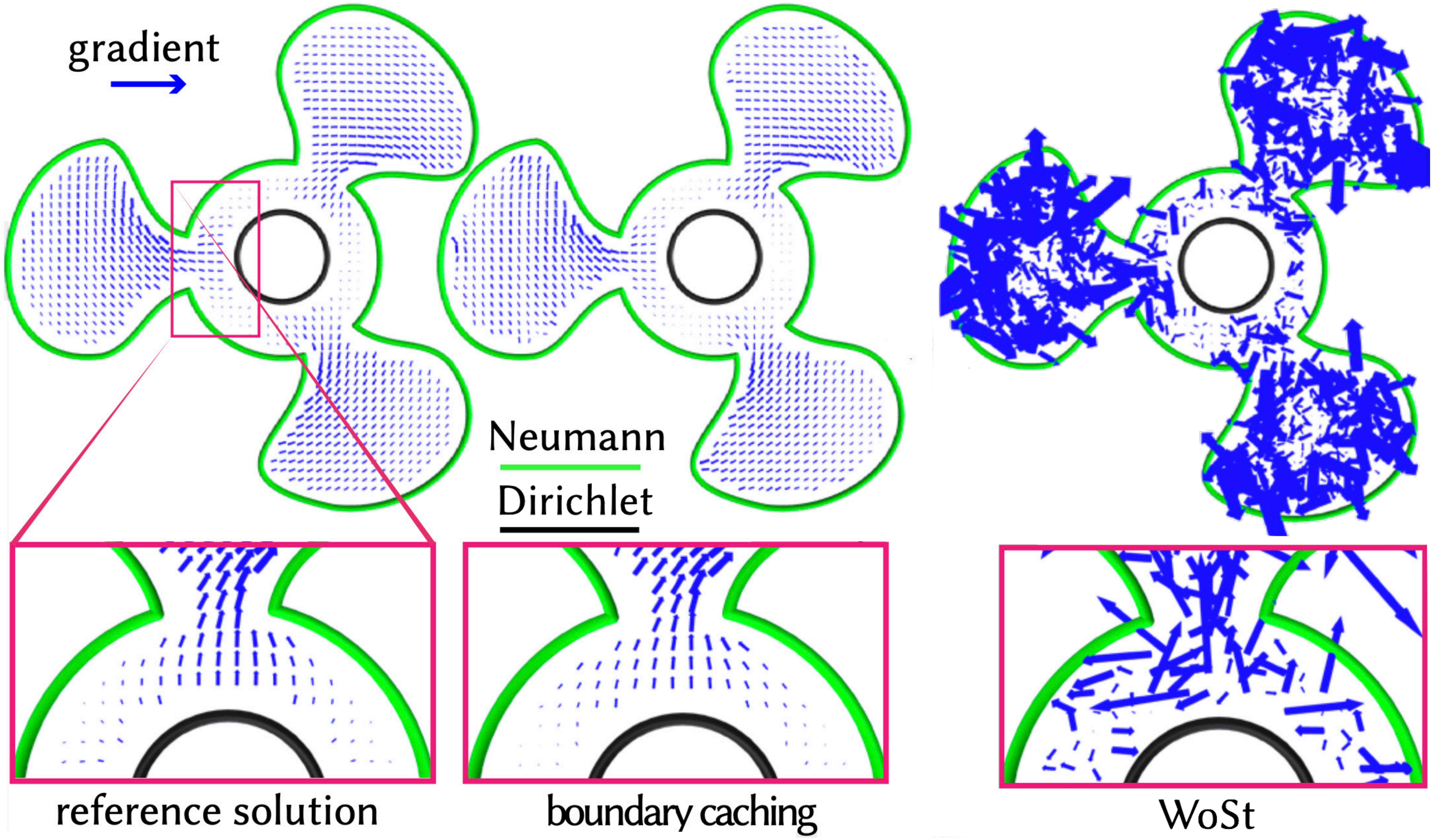
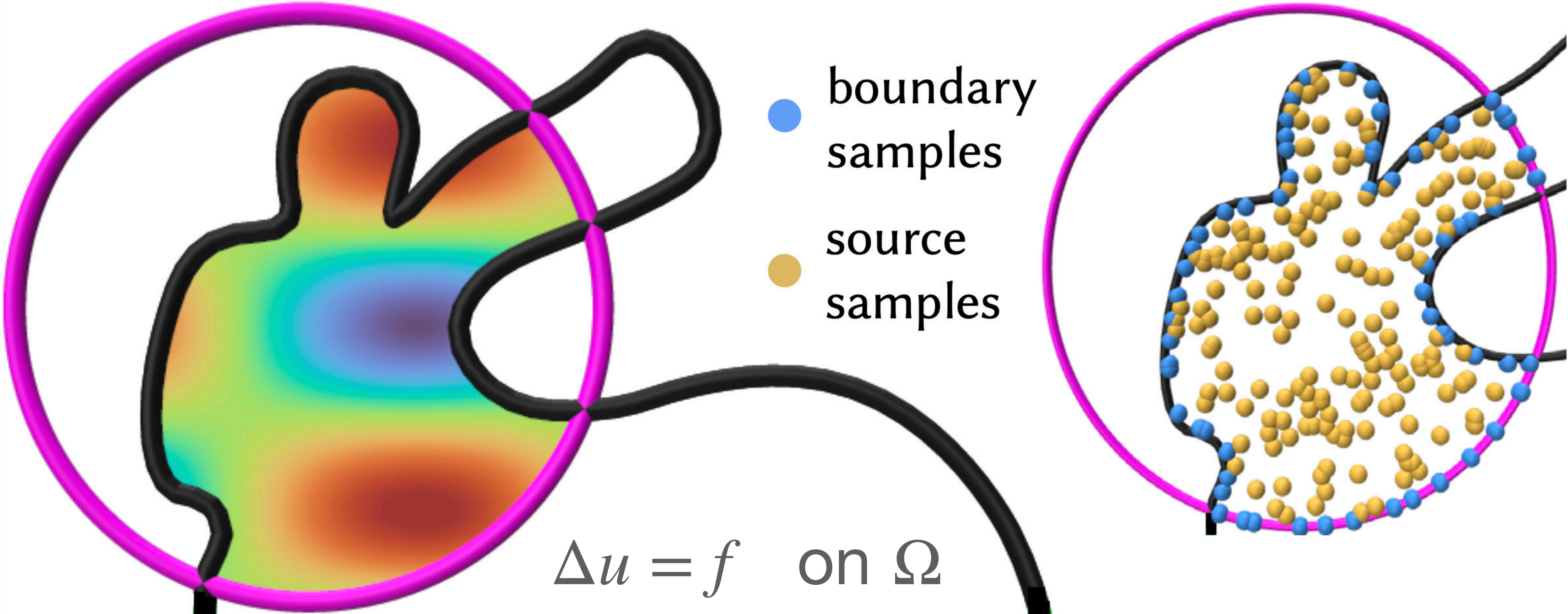


Source Term

Generate cache samples for source values f **inside** domain:

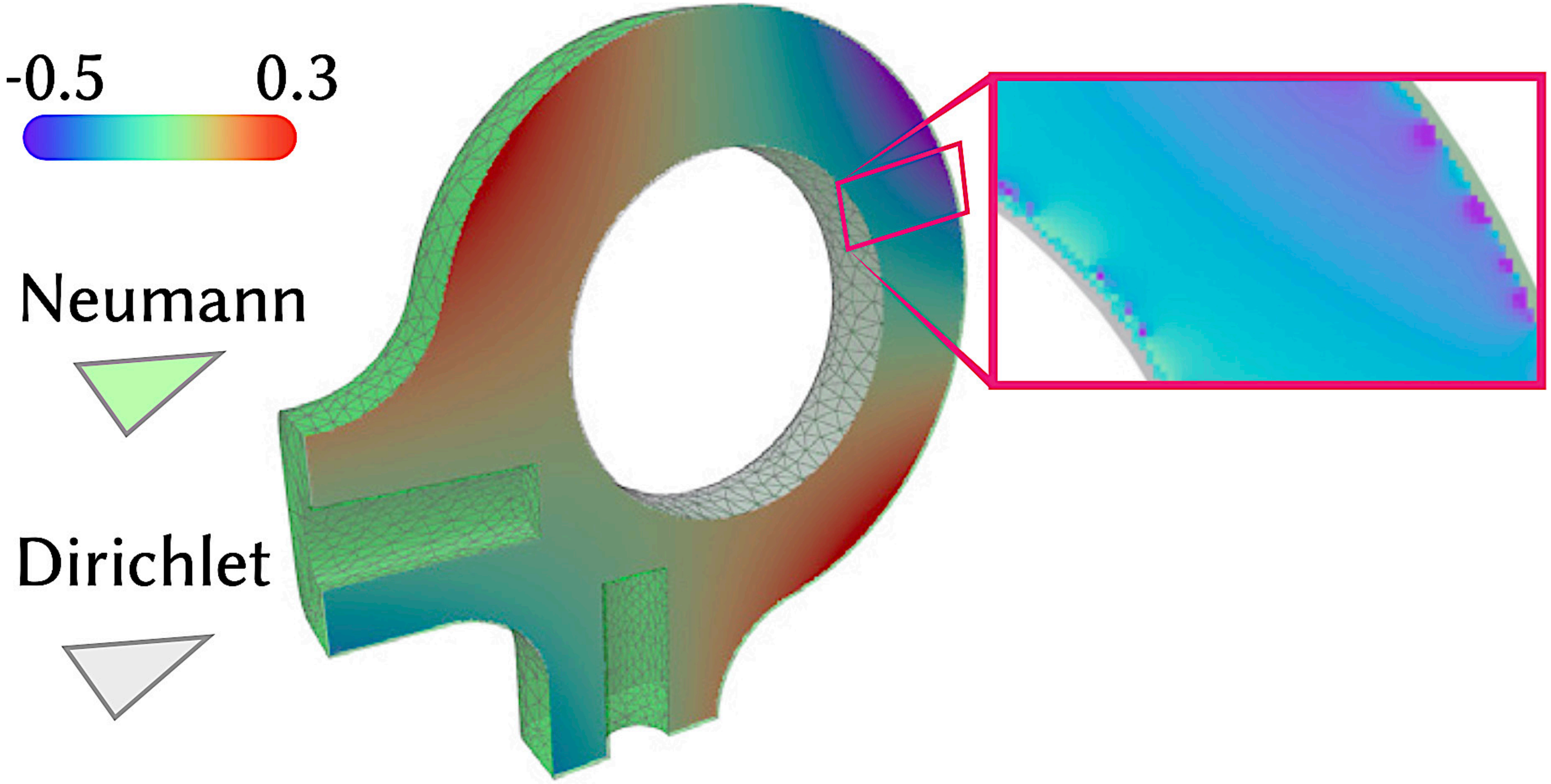
no random walks needed



$$\Delta u = f \quad \text{on } \Omega$$

Singularities

Artifacts near the boundary due to **lack of importance sampling**



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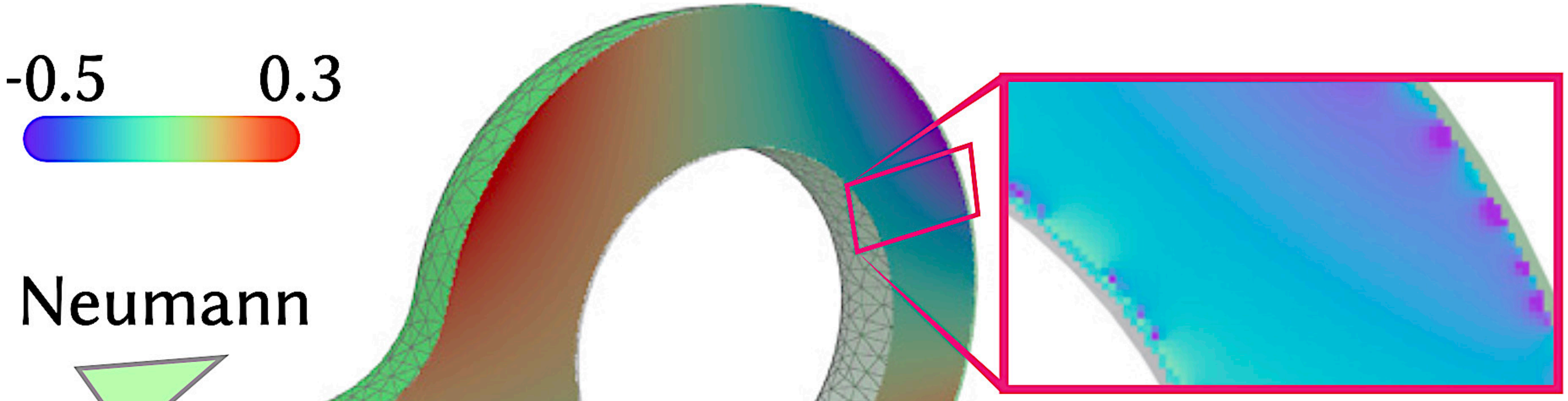
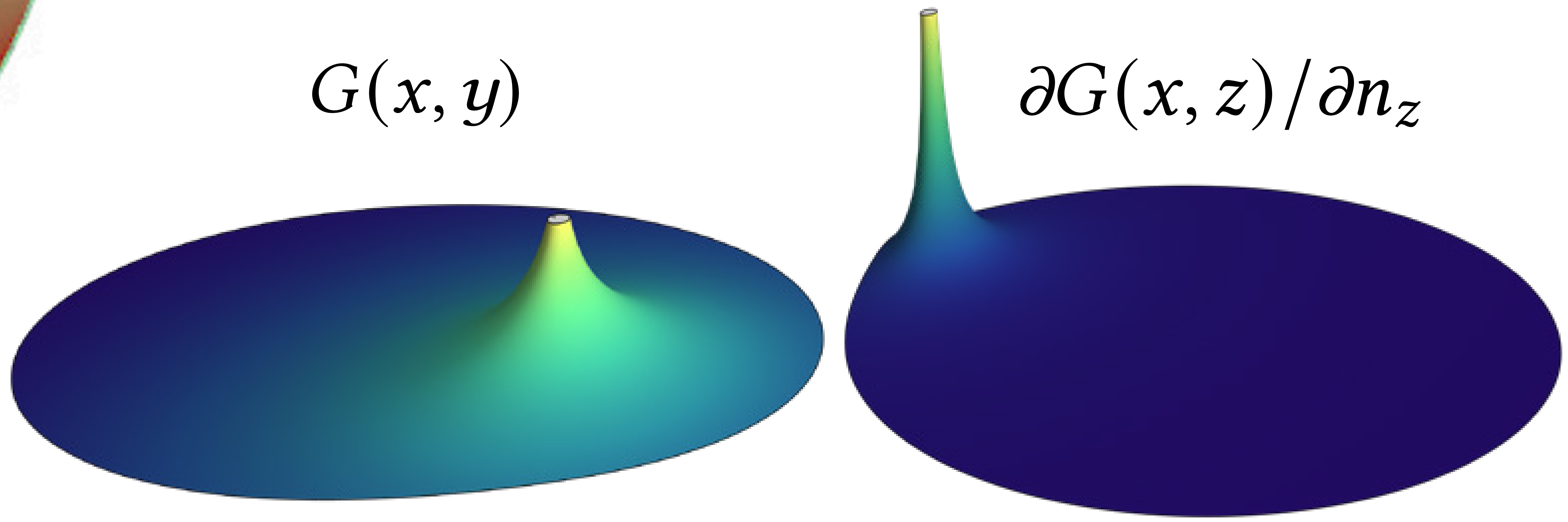
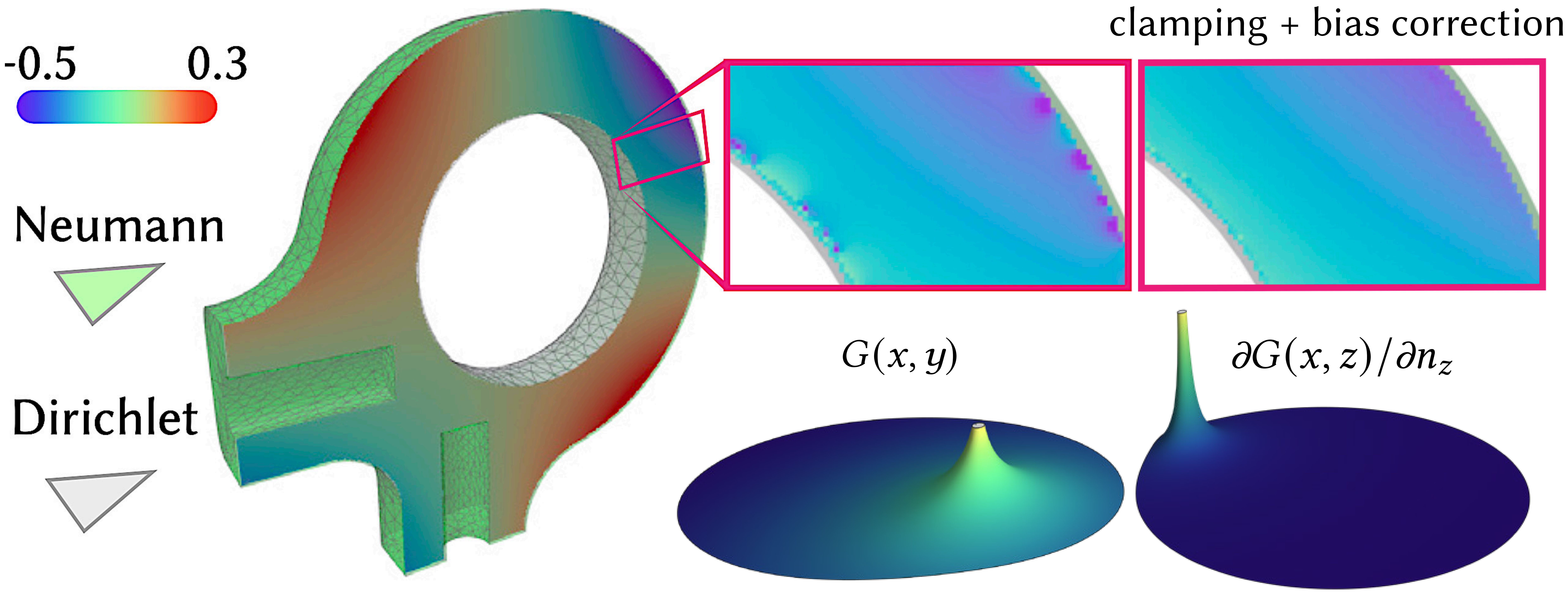




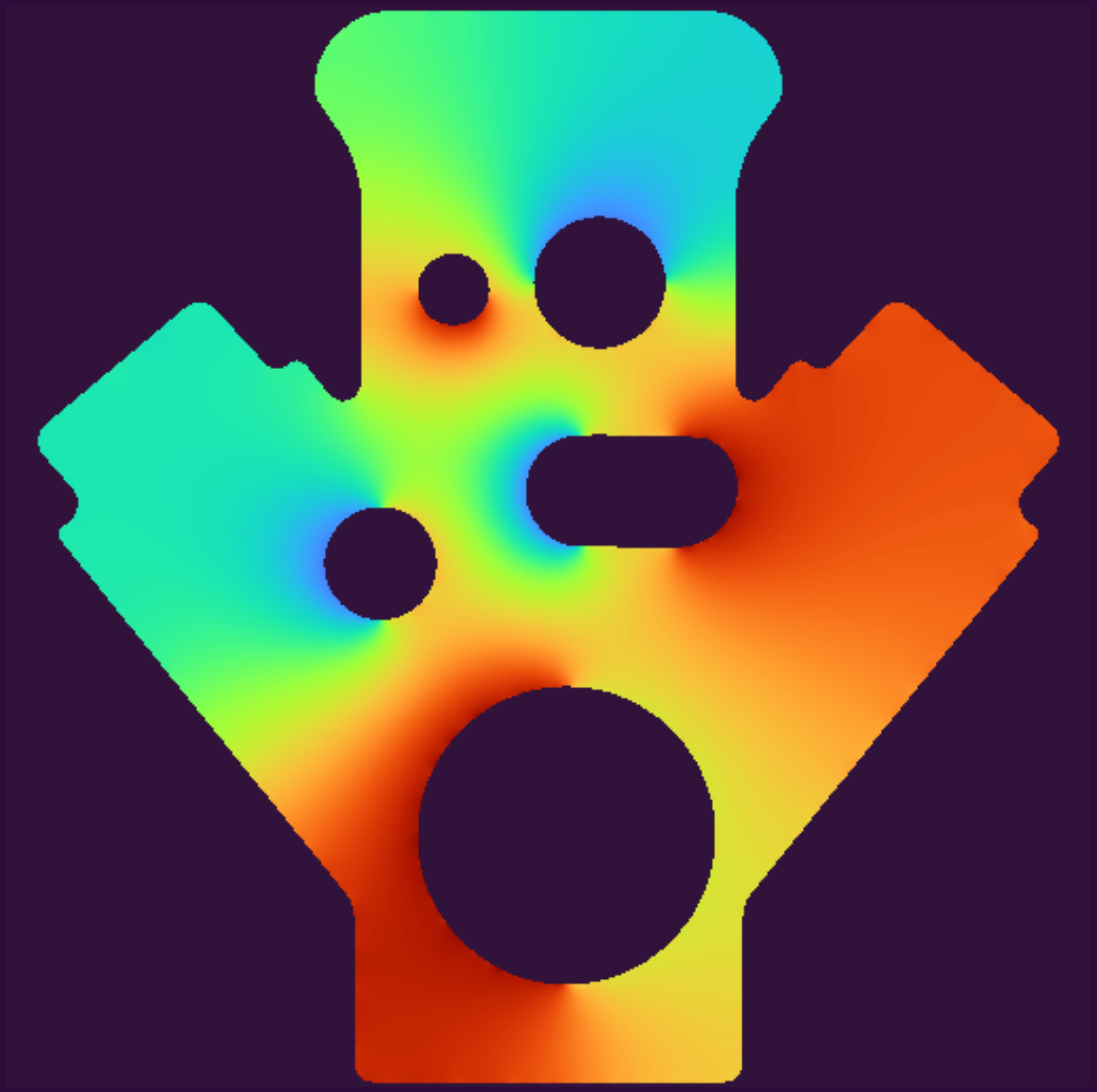
Singularities

Artifacts near the boundary due to **lack of importance sampling**

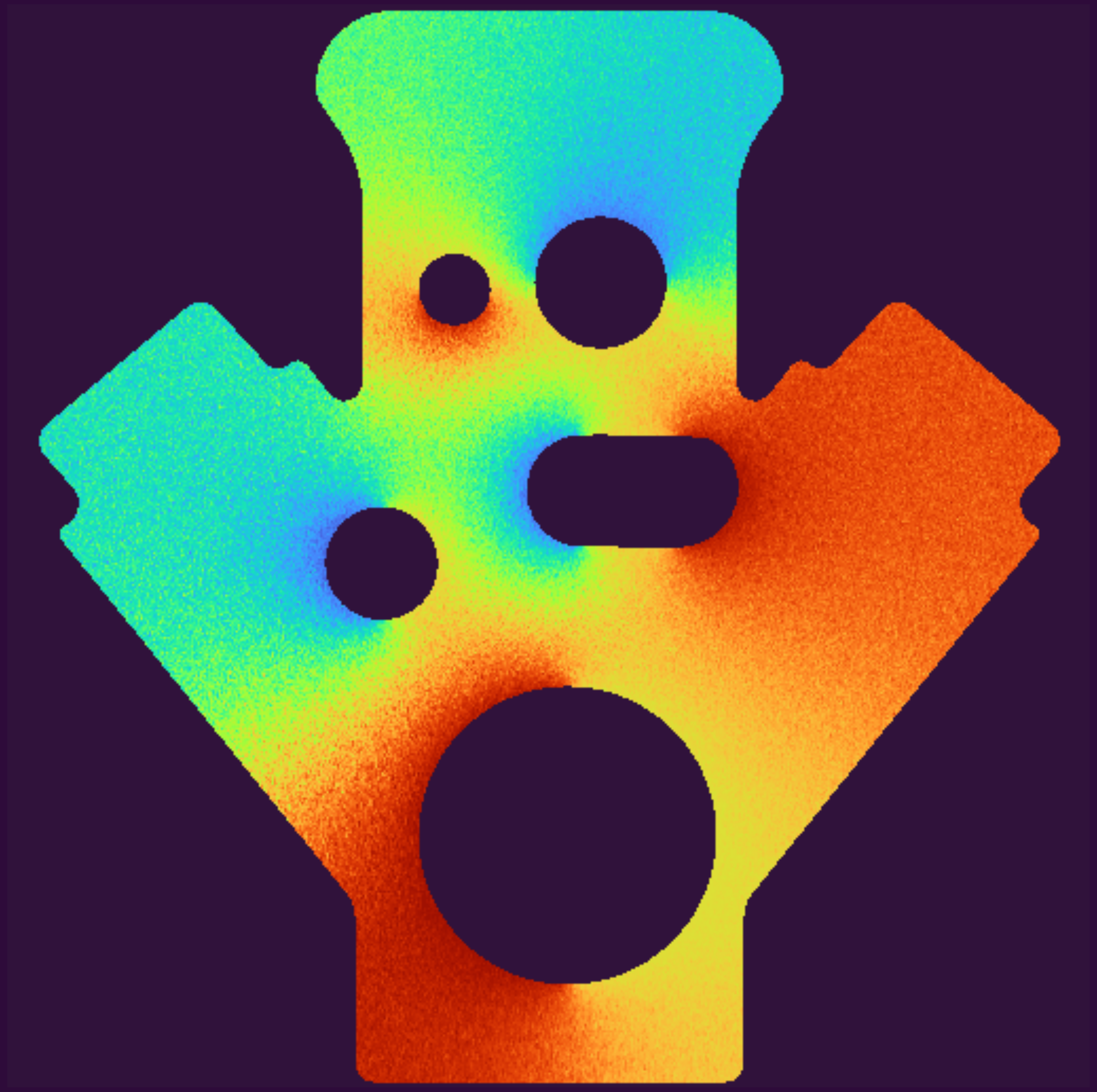


VALIDATION & COMPARISONS

Benefits of BVC

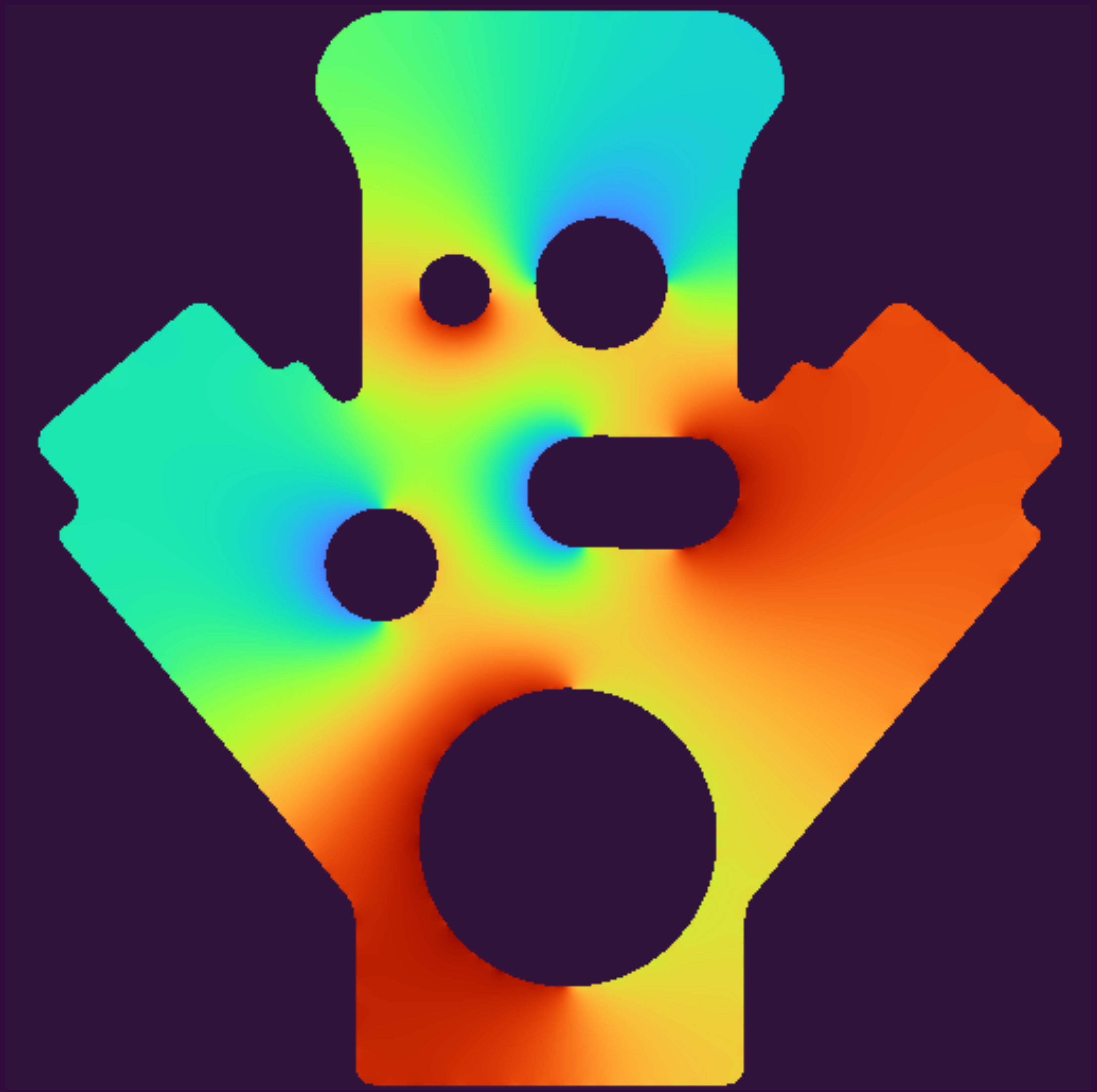


Boundary Value Caching



Walk on Stars

Benefits of BVC



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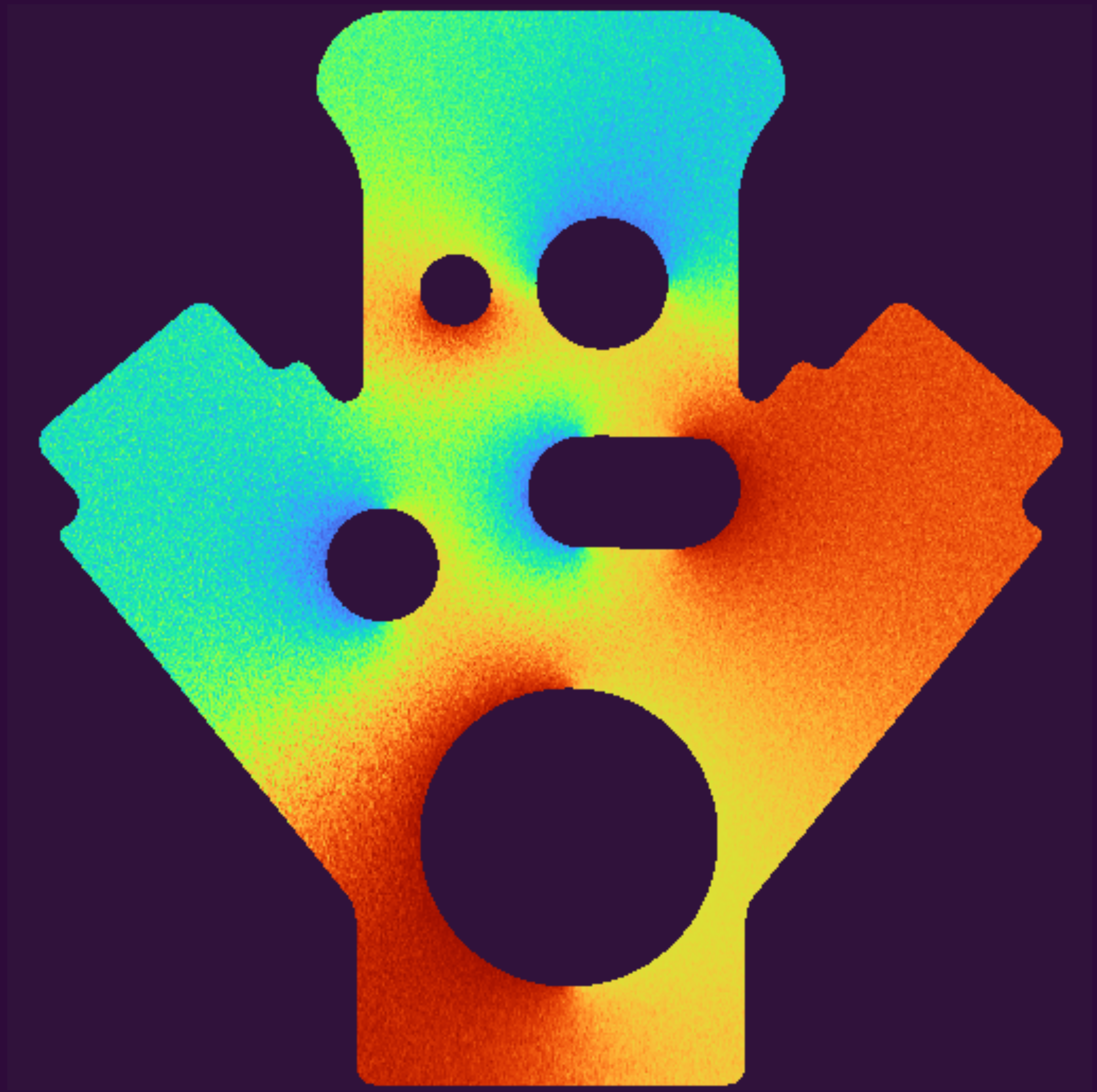
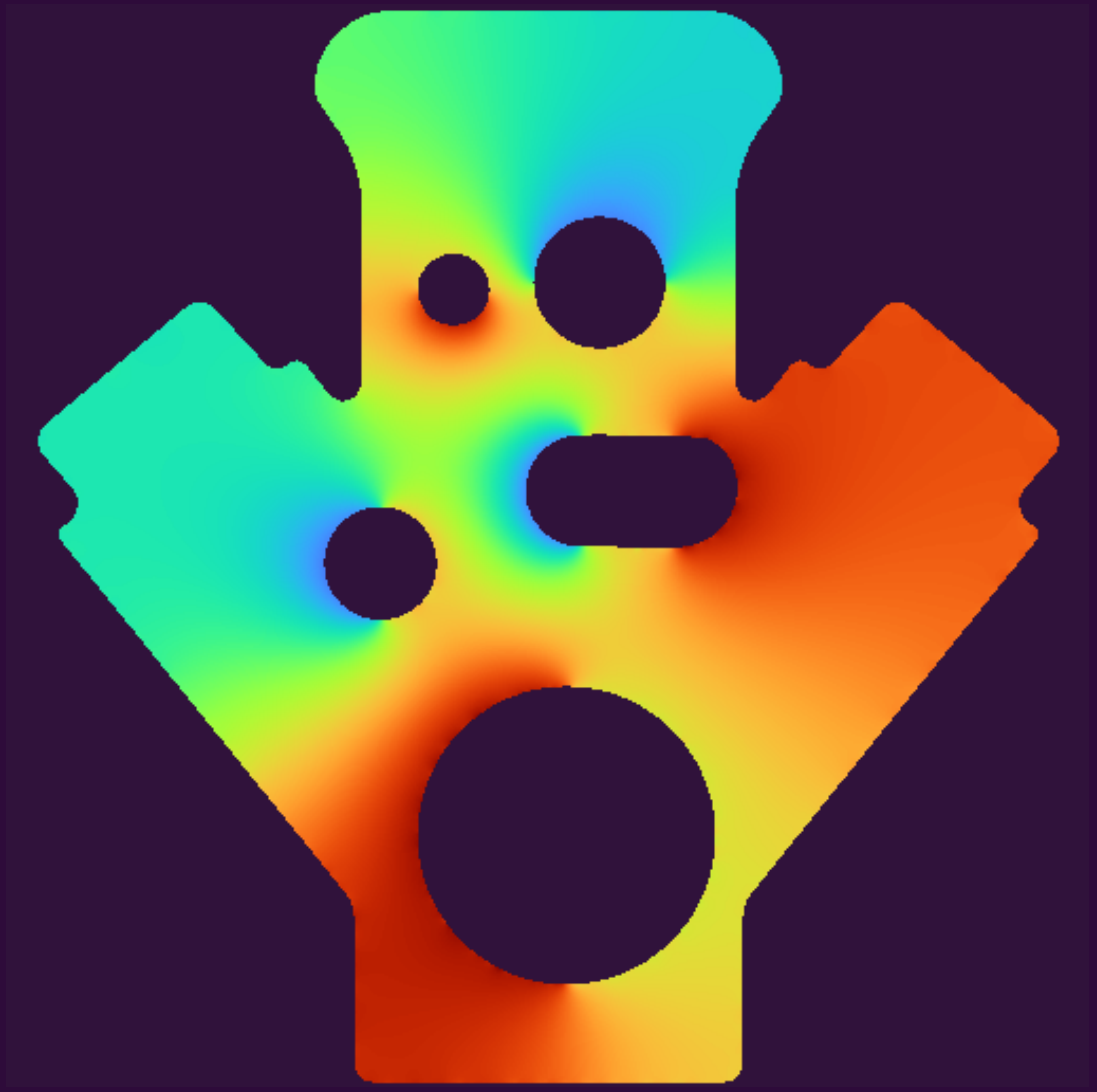


Walk on Stars

Improved run-time efficiency (sharing global information)

Benefits of BVC



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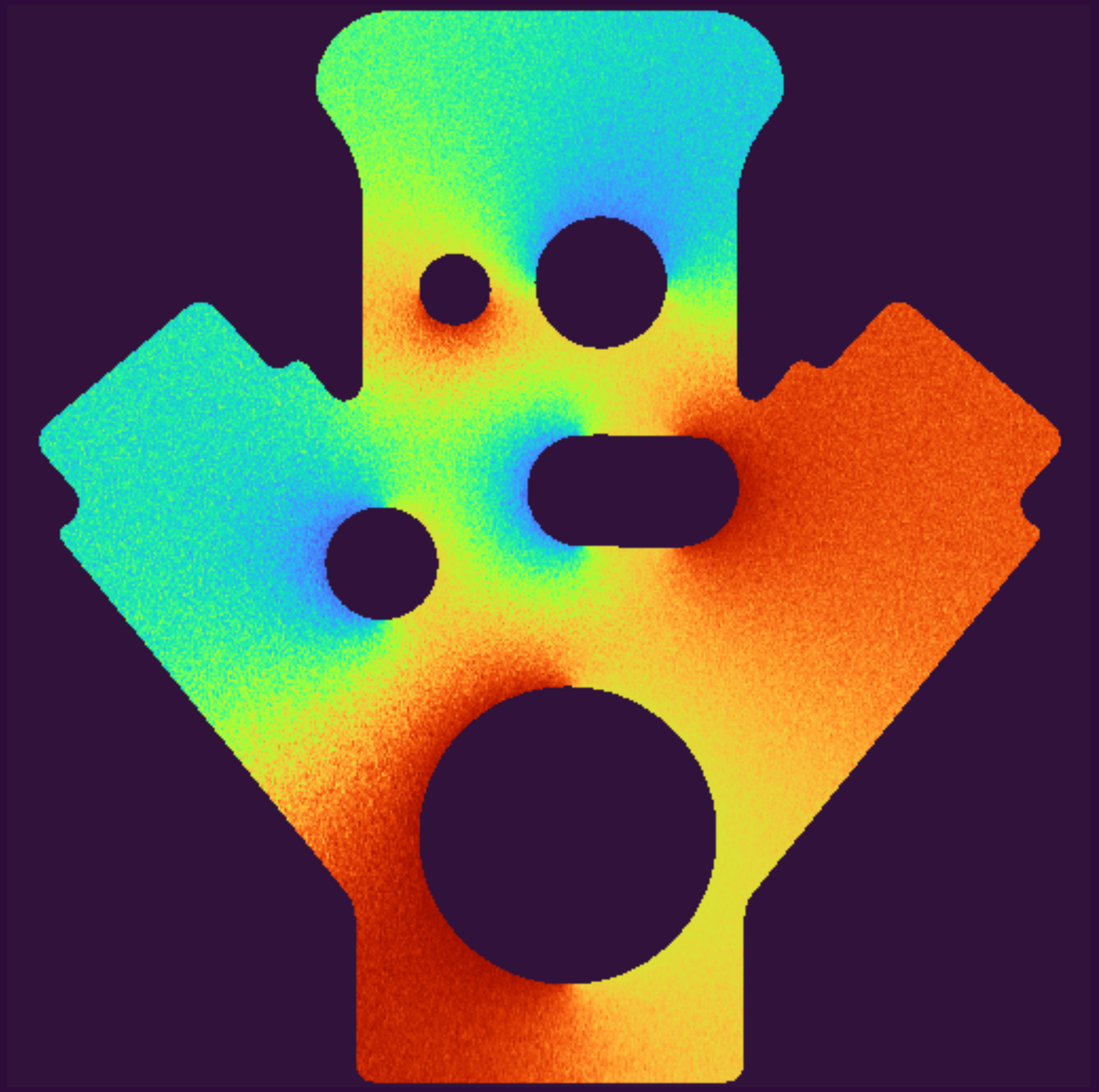
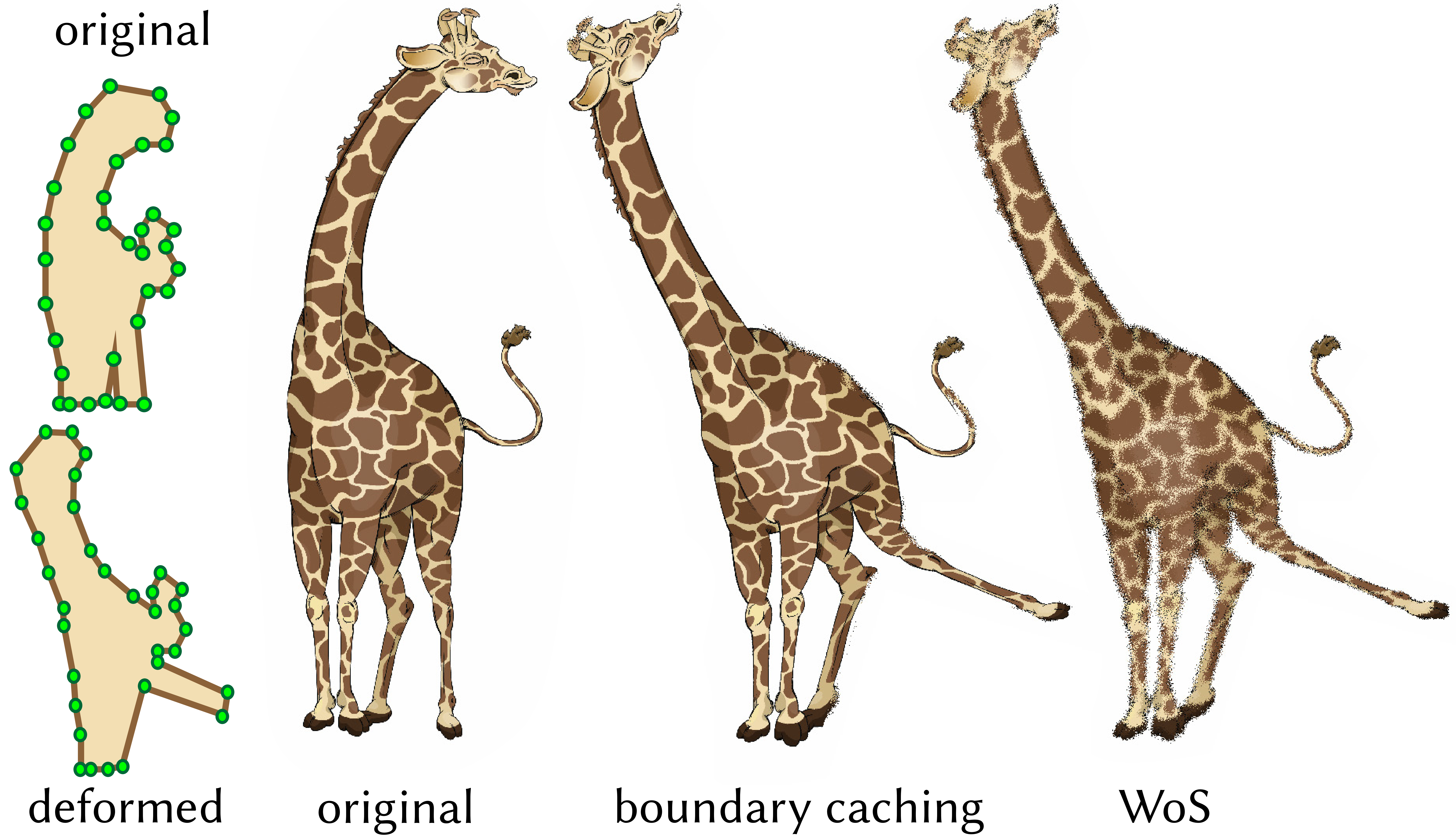


Walk on Stars

Improved run-time efficiency (sharing global information)

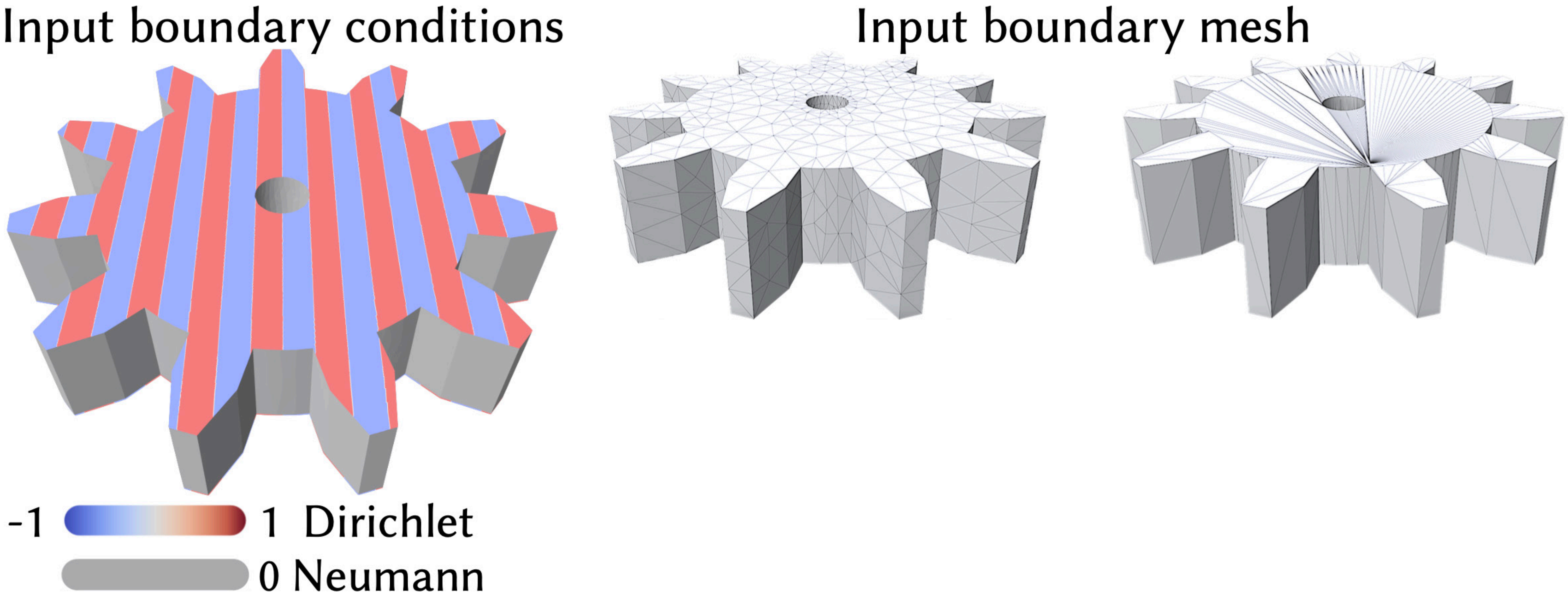
Suppressed noise (due to correlation)

Harmonic Interpolation of Texture Coordinates



Comparison to Boundary Element Method

Like WoSt, BVC is **not** affected by quality of discretization



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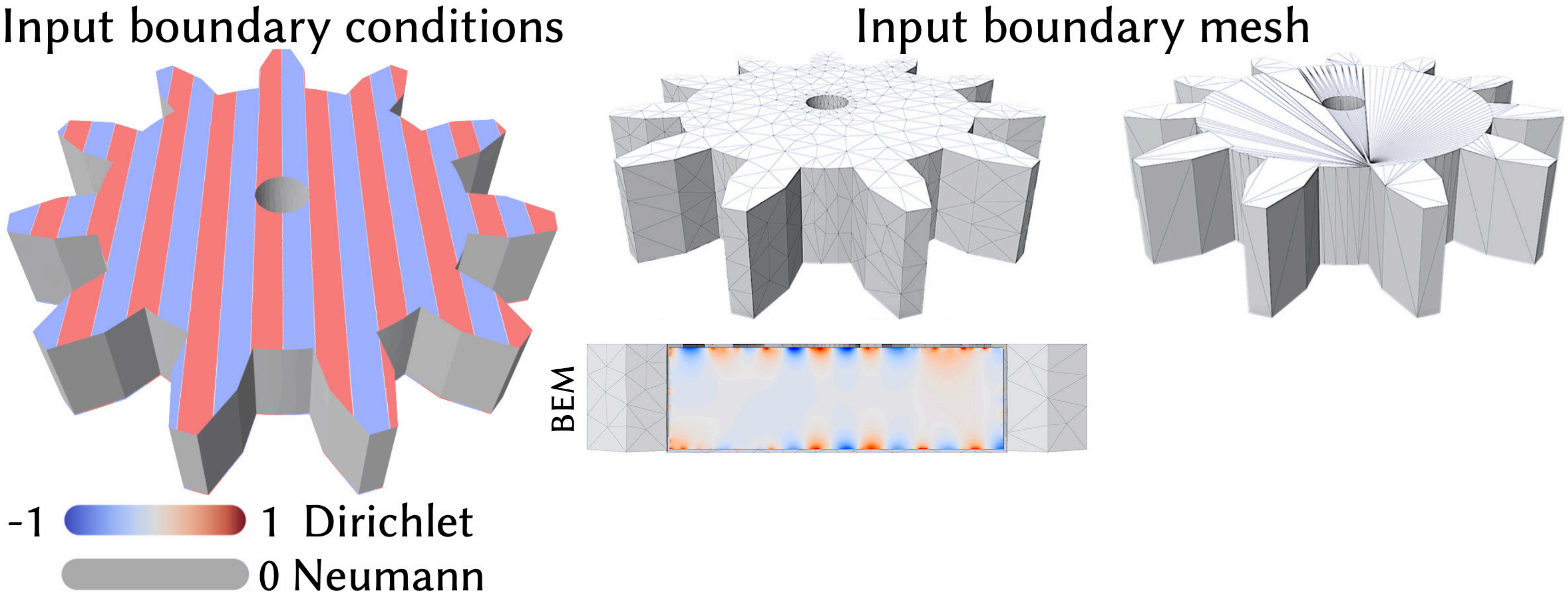

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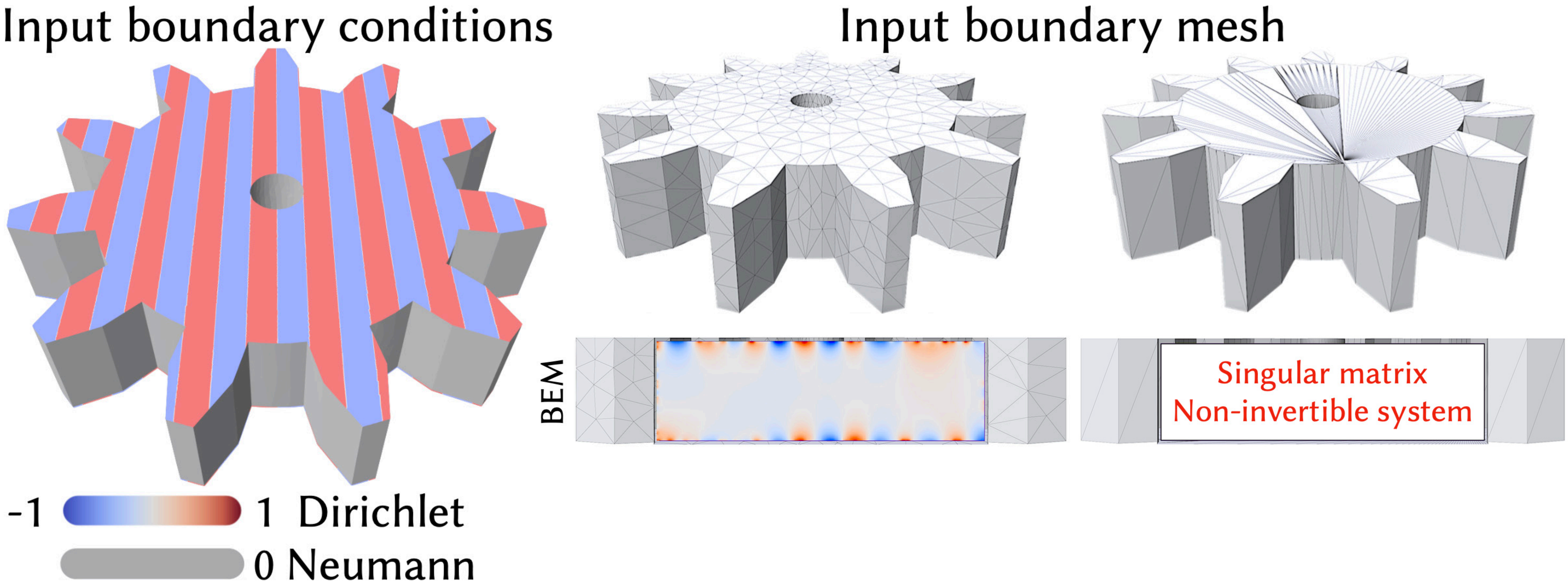

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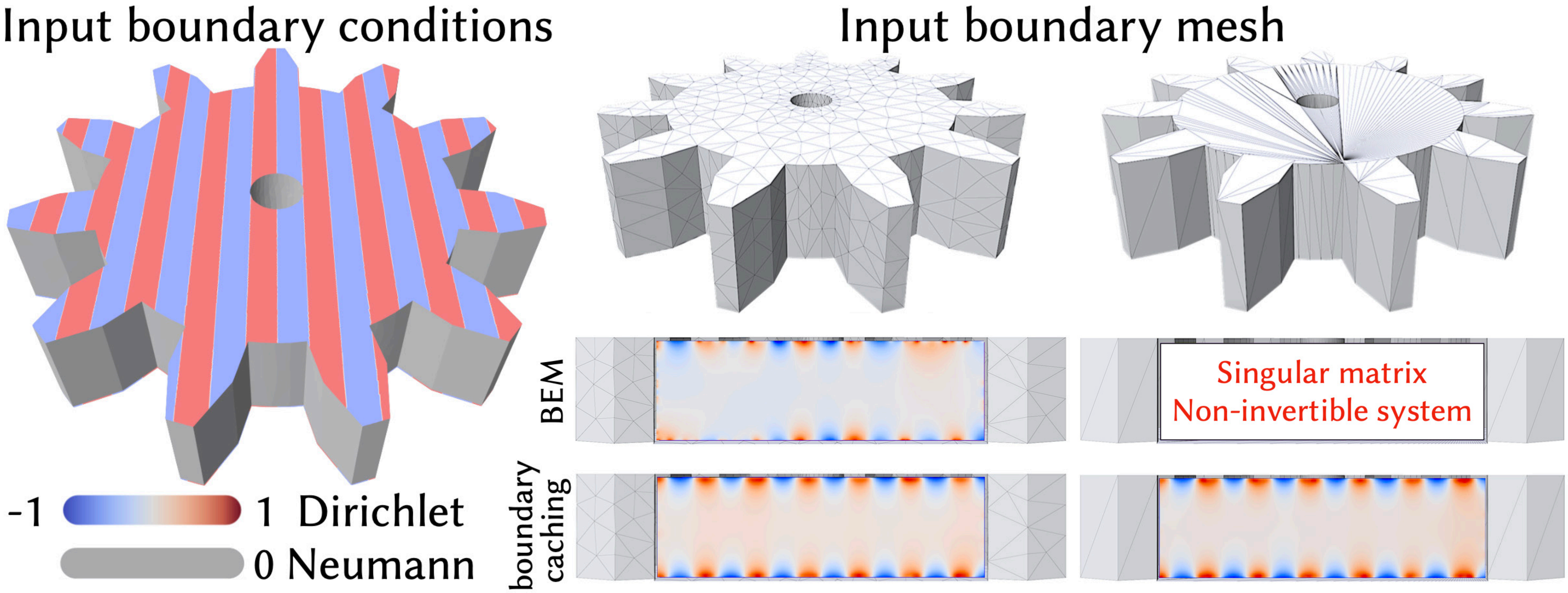
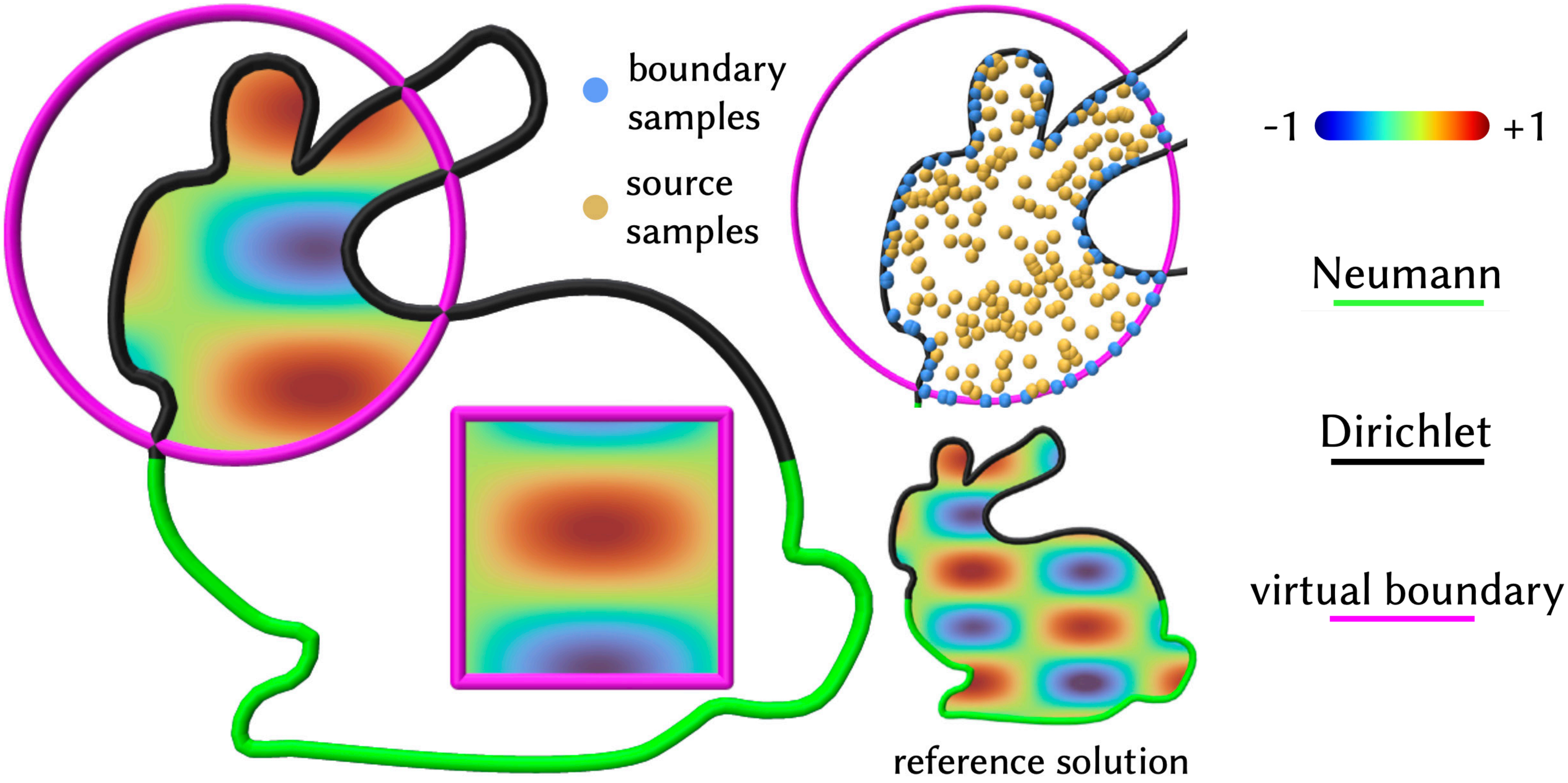
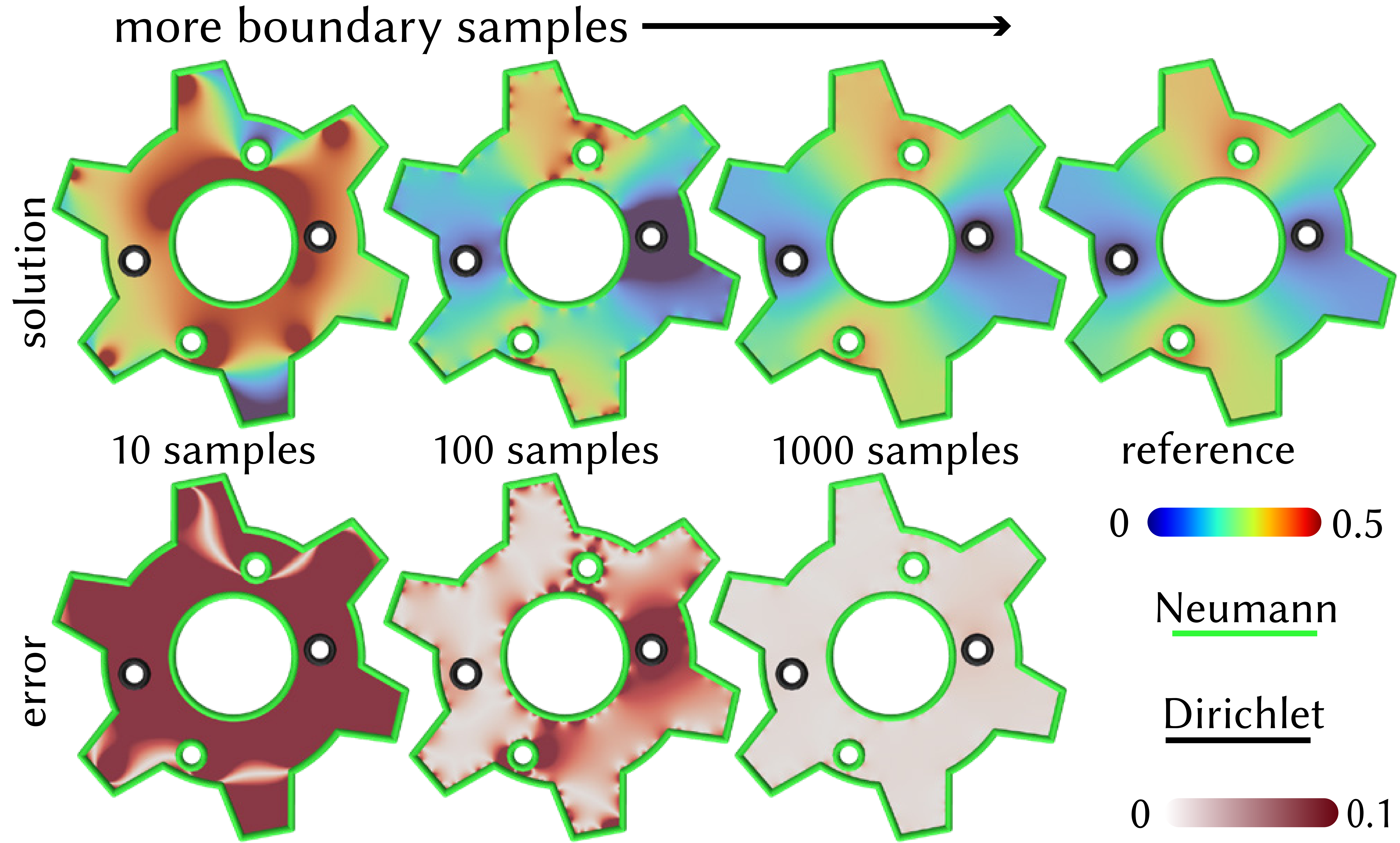


Output Sensitivity with BVC

Can focus computation in **local regions of interest**

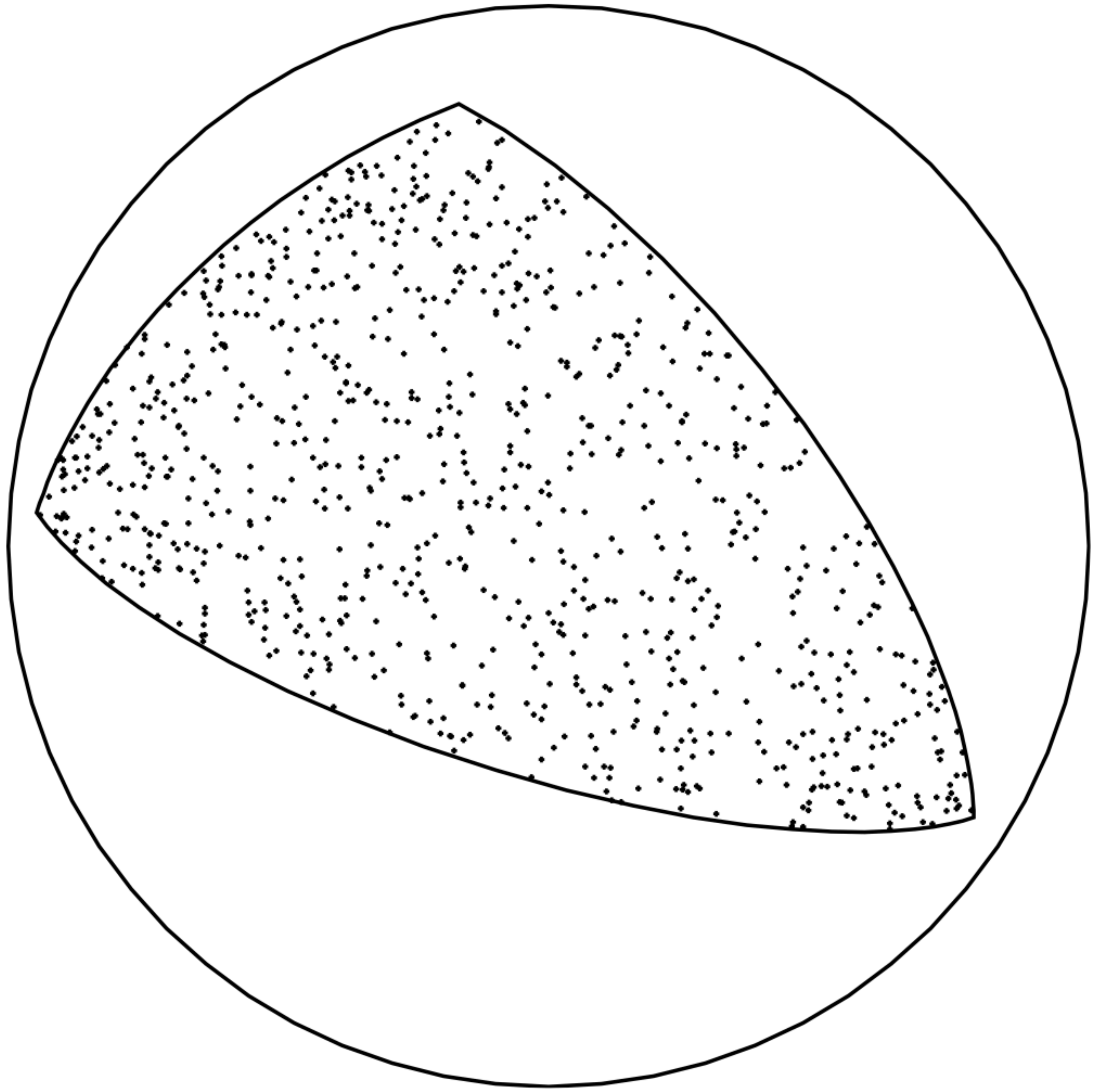


Error and Convergence



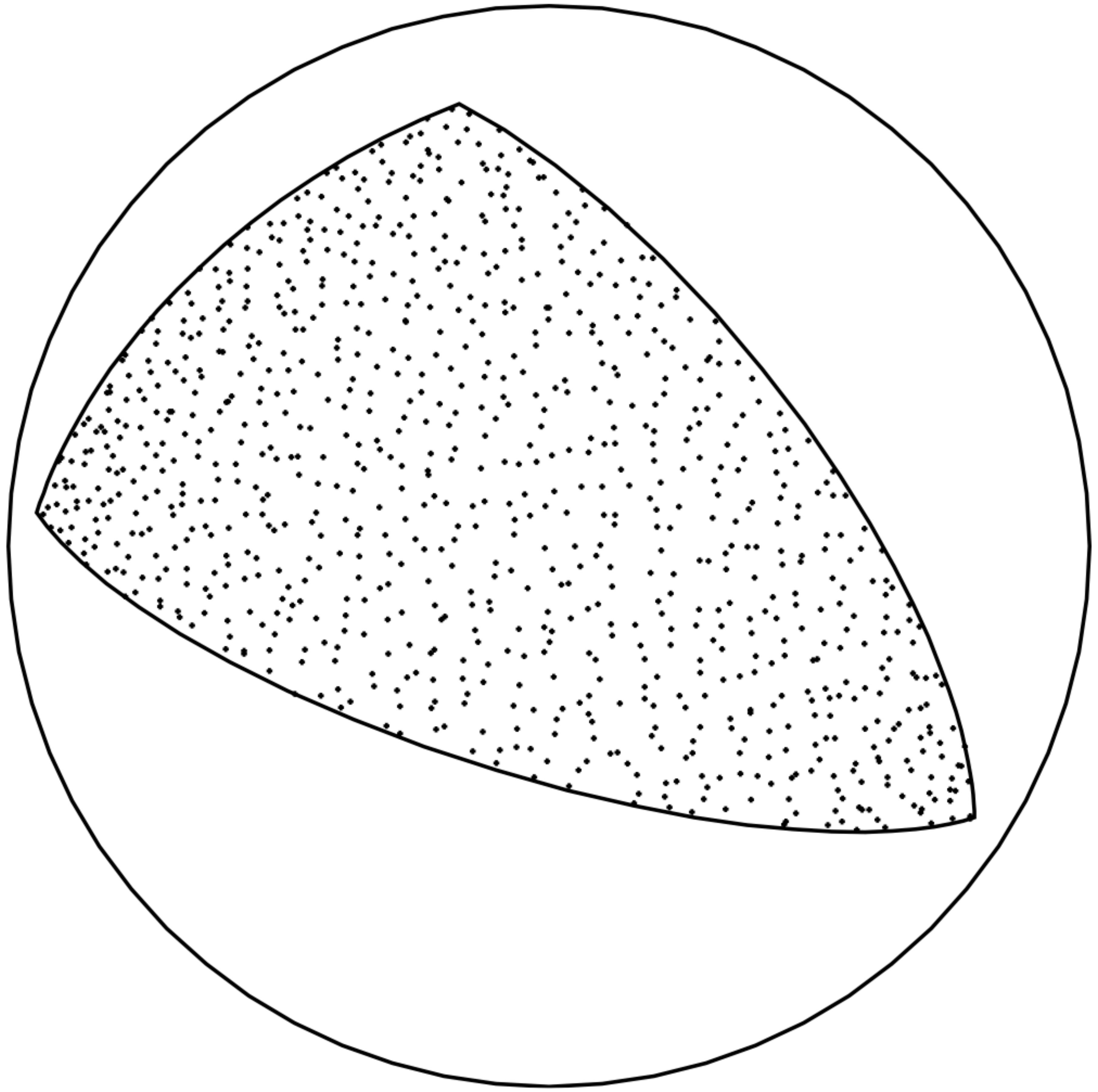
Stratification

uniform



"more clumpy"

stratified



"more even"

FUTURE WORK

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Any improvements to WoSt benefit BVC!

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Principled approach to estimate $\frac{du}{dn}$ on $\partial\Omega_D$

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Splatting has quadratic complexity:

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Reuse information ***during*** *a* walk?

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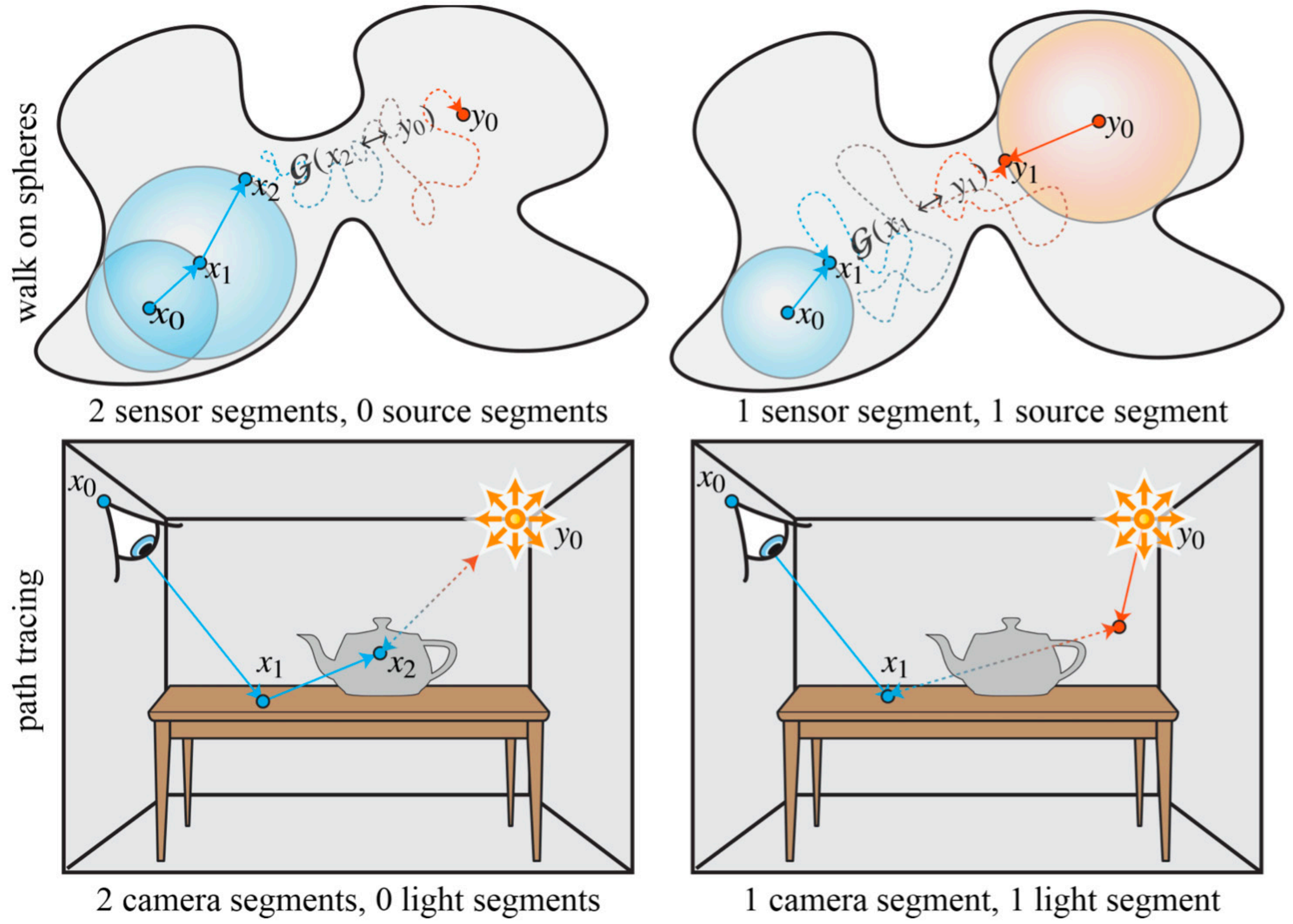
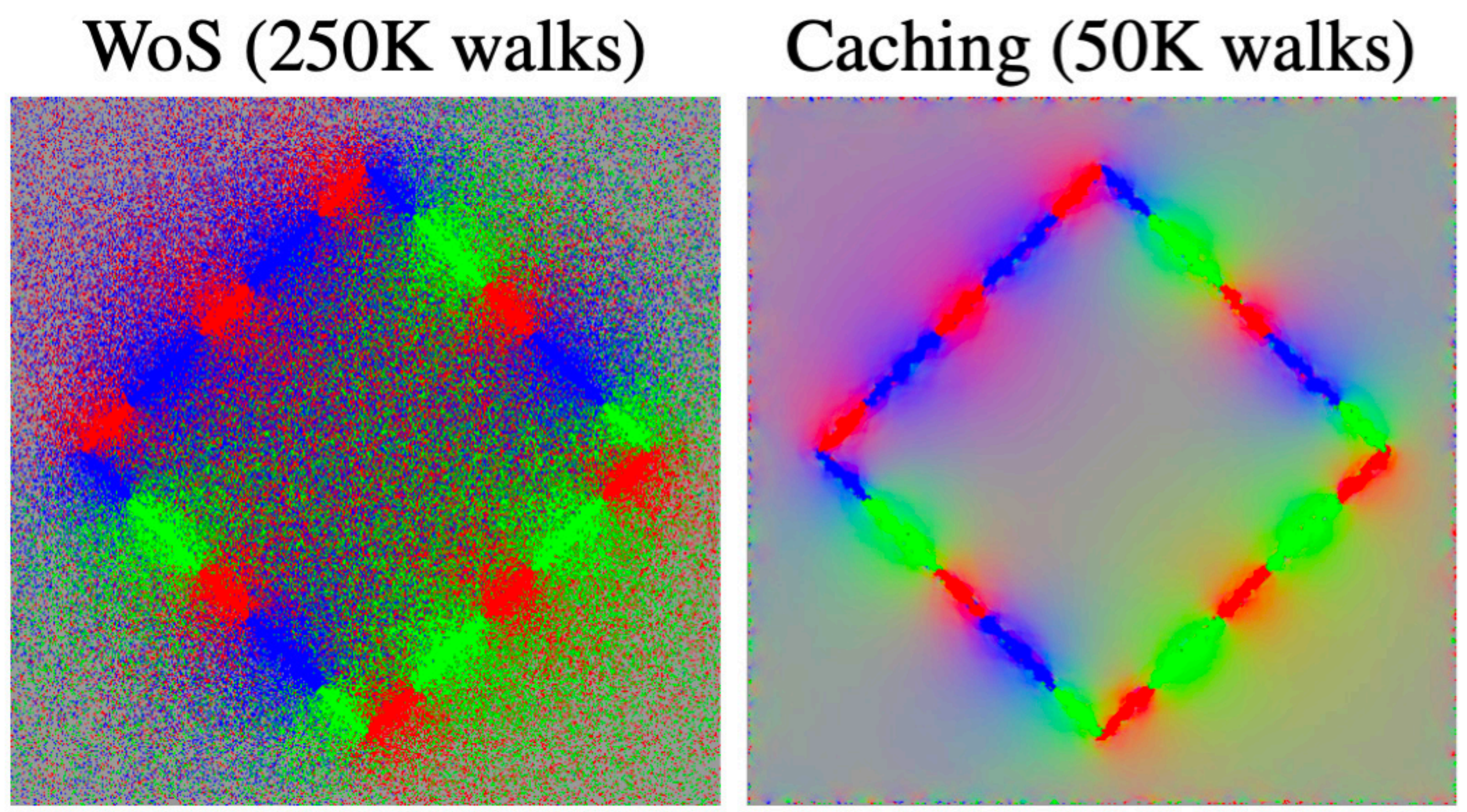
Reuse information ***during*** *a* walk?

Unified caching w BVC, MVC, Bidirectional WoS?



[Qi, Seyb, Bitterli, Jarosz, EGSR 2022]

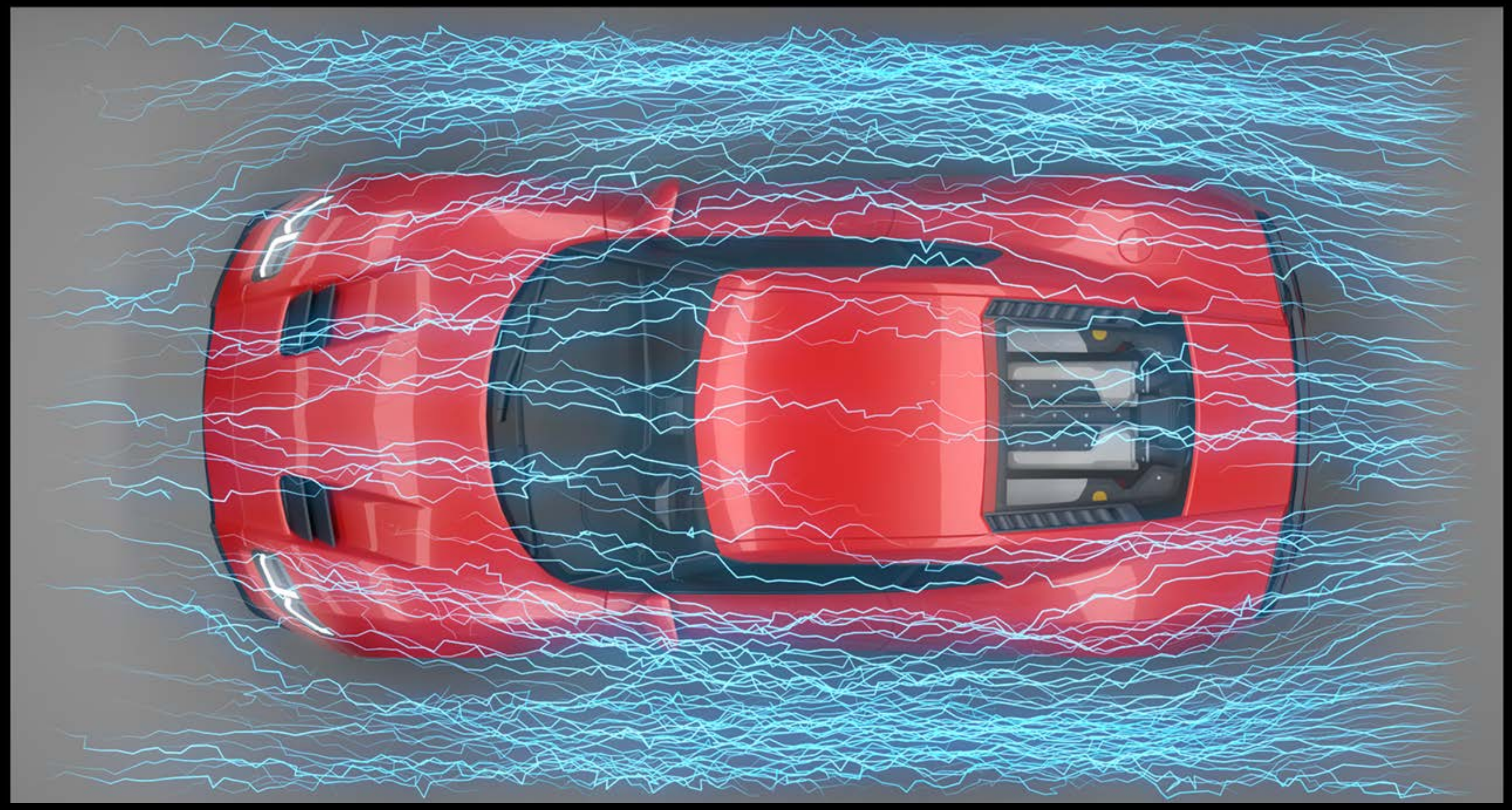


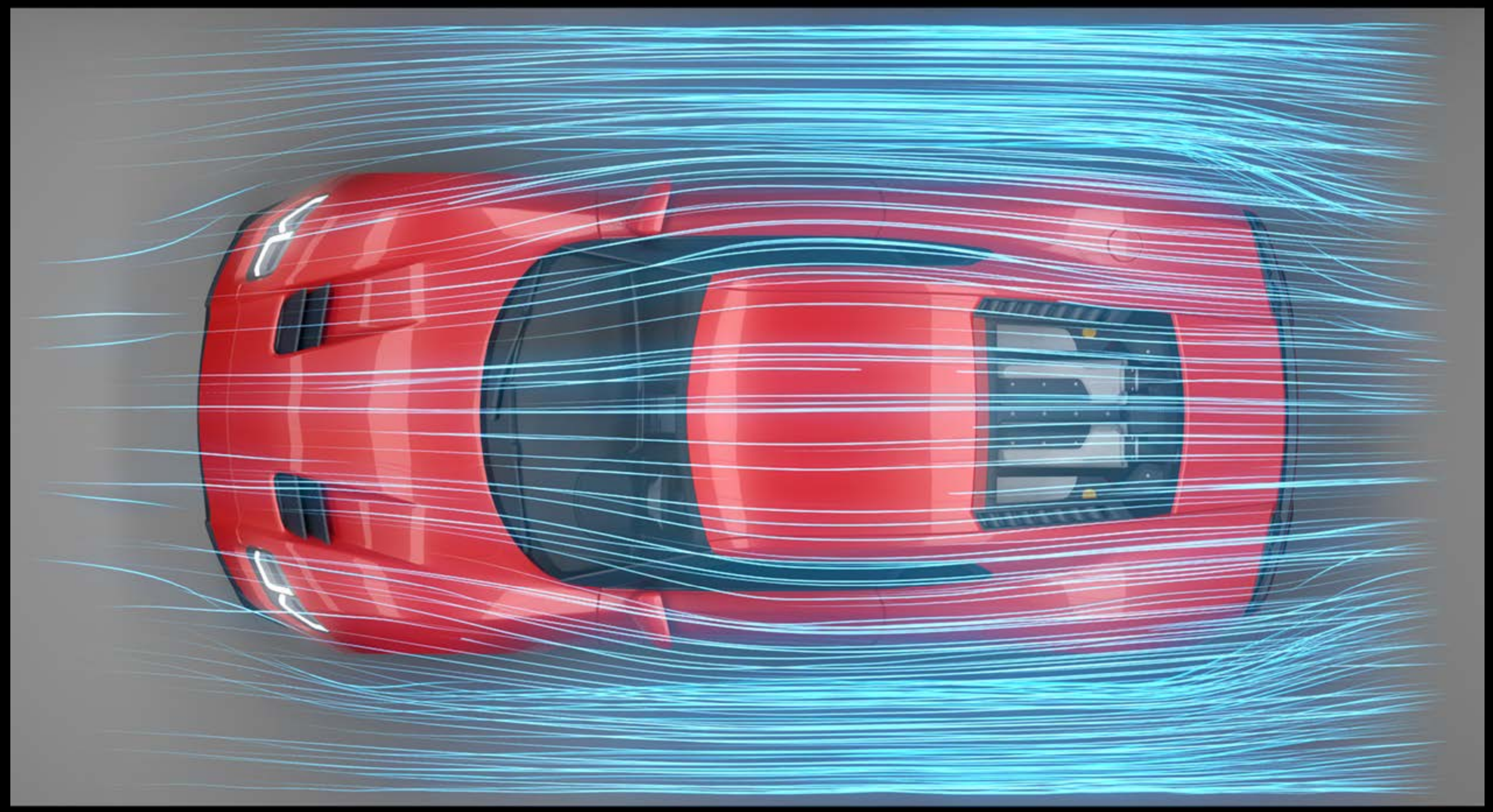
[Bakbouk & Peers, EGSR 2023]

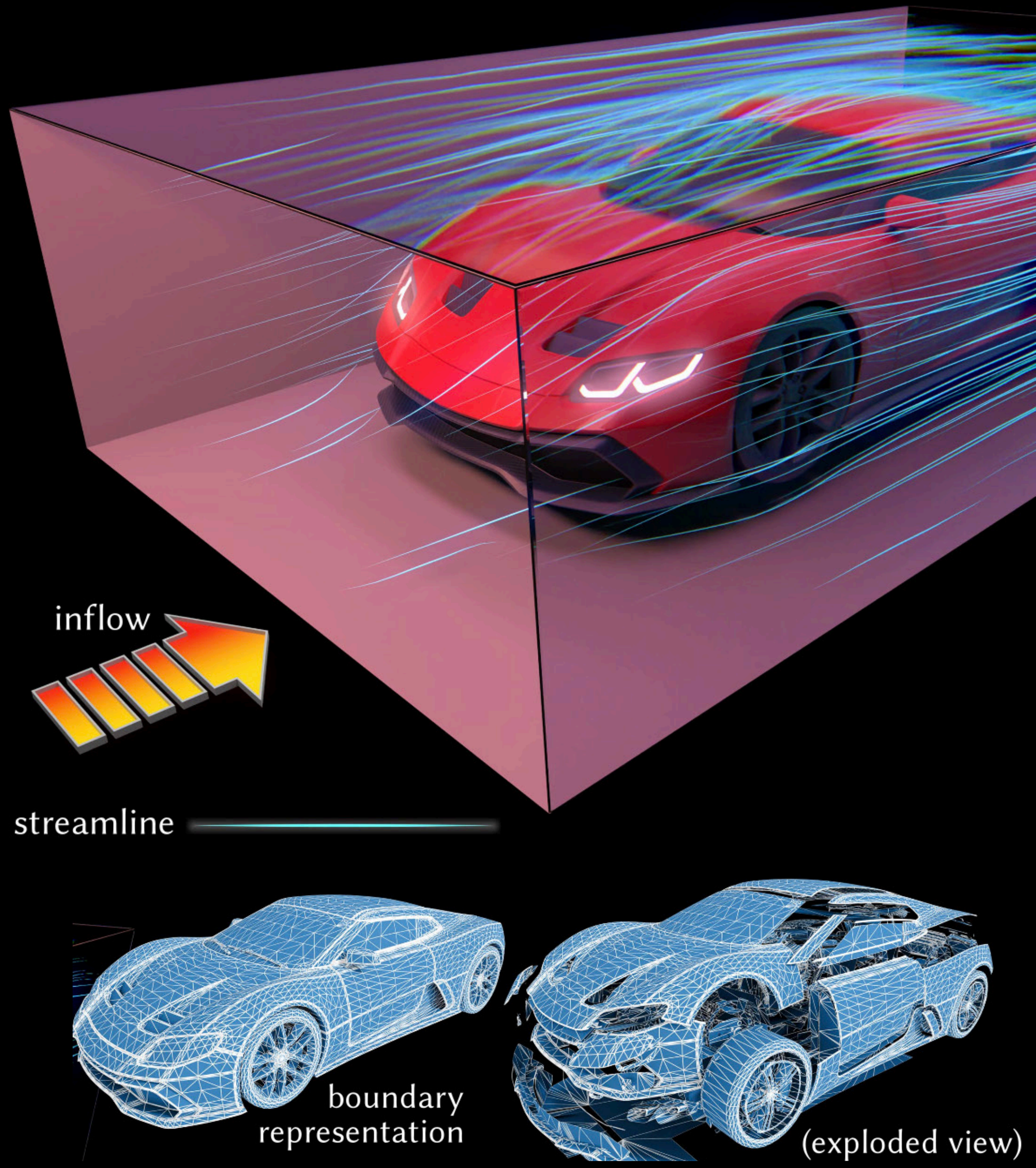
paper

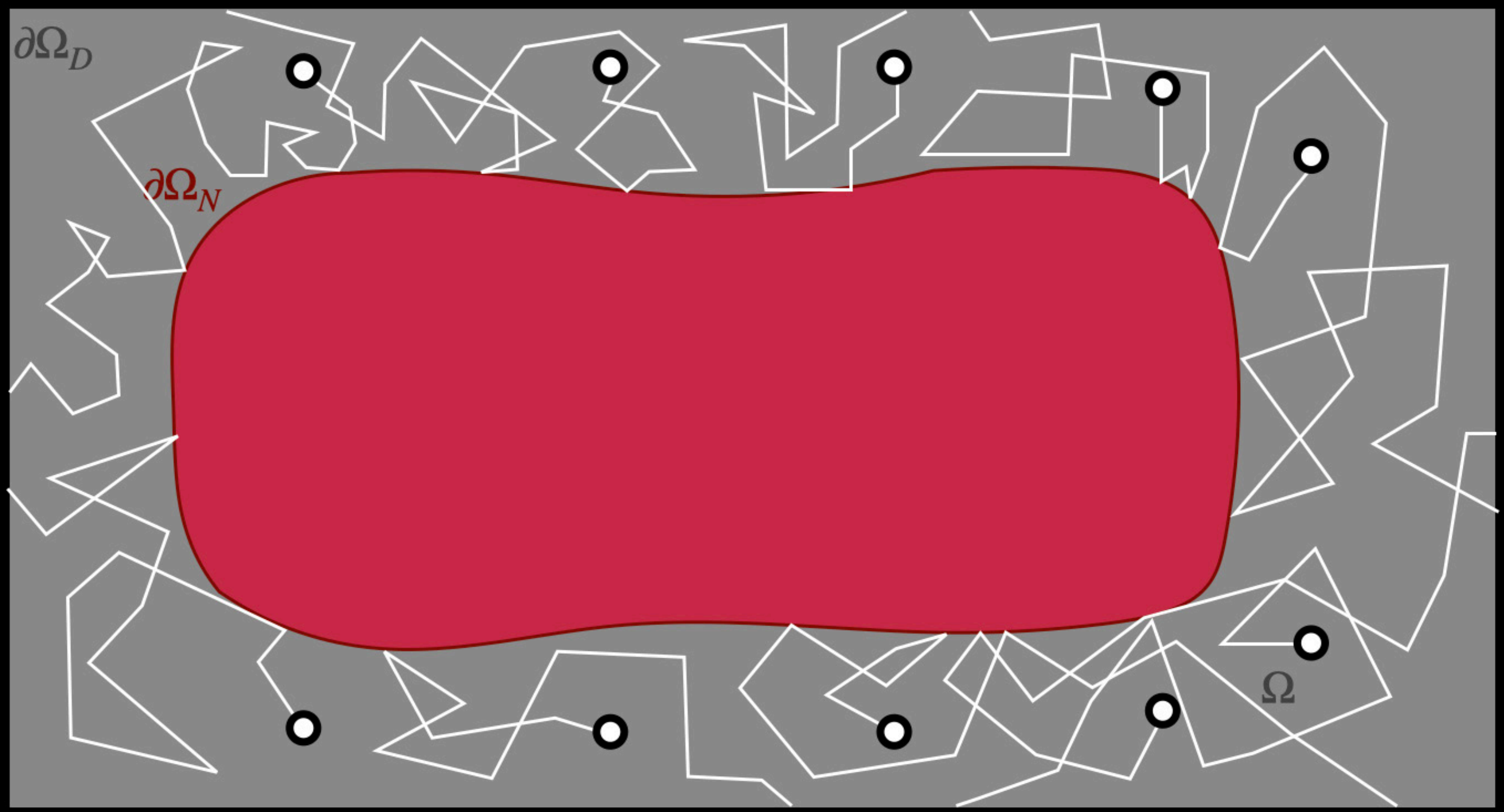
code

Thank you!

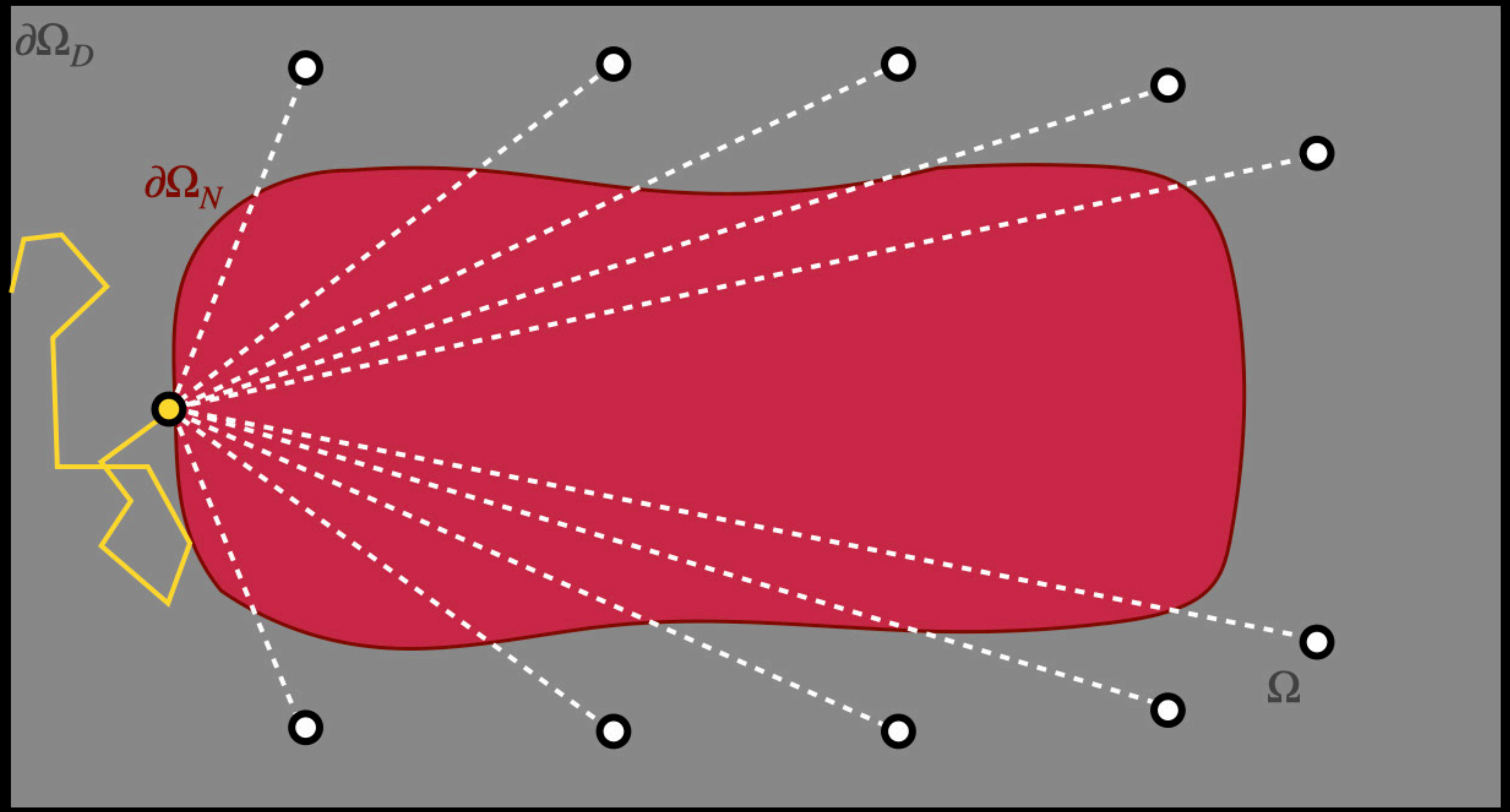








pointwise



BVC